A moduli curve for compact conformally-Einstein Kähler manifolds

A. Derdzinski and G. Maschler

Abstract

We classify quadruples (M,g,m,τ) in which (M,g) is a compact Kähler manifold of complex dimension $m>2$ with a nonconstant function τ on M such that the conformally related metric g/τ^2, defined wherever $\tau\neq 0$, is Einstein. It turns out that M then is the total space of a holomorphic $\mathbf{C}\mathrm{P}^1$ bundle over a compact Kähler-Einstein manifold (N,h). The quadruples in question constitute four disjoint families: one, well-known, with Kähler metrics g that are locally reducible; a second, discovered by Bérard Bergery (1982), and having $\tau\neq 0$ everywhere; a third one, related to the second by a form of analytic continuation, and analogous to some known Kähler surface metrics; and a fourth family, present only in odd complex dimensions $m\geqslant 9$. Our classification uses a *moduli curve*, which is a subset $\mathcal{C}$, depending on m, of an algebraic curve in $\mathbf{R}^2$. A point (u,v) in $\mathcal{C}$ is naturally associated with any (M,g,m,τ) having all of the above properties except for compactness of M, replaced by a weaker requirement of "vertical" compactness. One may in turn reconstruct M,g and τ from this (u,v) coupled with some other data, among them a Kähler-Einstein base (N,h) for the $\mathbf{C}\mathrm{P}^1$ bundle M. The points (u,v) arising in this way from (M,g,m,τ) with compact M form a countably infinite subset of $\mathcal{C}$.

0. Introduction

This paper may be treated as a sequel to [10] – [11], and provides a classification, up to biholomorphic isometries, of compact Kähler manifolds in complex dimensions $m>2$ that are almost-everywhere conformally Einstein. Specifically, we describe all quadruples (M,g,m,τ) in which

$$\begin{aligned}&M \text{ is a compact complex manifold of complex dimension } m\geqslant 3 \text{ with a}\\ &\text{Kähler metric } g \text{ and a nonconstant } C^\infty \text{ function } \tau: M\to\mathbf{R} \text{ such that the}\\ &\text{conformally related metric } \tilde g = g/\tau^2, \text{ defined wherever } \tau\neq 0, \text{ is Einstein.}\end{aligned}\tag{0.1}$$

When $m=2$, we also classify quadruples (M,g,m,τ) with the following property.

$$\begin{aligned}&\text{Condition (0.1) holds except for the requirement that } m\geqslant 3, \text{ now}\\ &\text{replaced by } m=2. \text{ In addition, } d\tau\wedge d\Delta\tau=0 \text{ everywhere in } M.\end{aligned}\tag{0.2}$$

Our classification of (0.1) – (0.2) is summarized in Theorems 1.3, 1.5 and 1.6. By Theorem 1.3, M must be the total space of a holomorphic $\mathbf{C}\mathrm{P}^1$ bundle over a compact Kähler-Einstein manifold (N,h) such that the metrics g and h make the bundle projection $M\to N$ a horizontally homothetic submersion [13] with totally geodesic fibres.

Theorem 1.3 also implies that every quadruple (M,g,m,τ) satisfying (0.1) or (0.2) belongs to one of the four disjoint families listed below, cf. Remark 1.4.

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The first and simplest family of examples with (0.1) or (0.2) involves *locally reducible* Kähler metrics g. They all have $\tau = 0$ somewhere in M. See §43.

By constructing the corresponding conformally-Kähler compact Einstein manifolds, Page [14] and Bérard Bergery [4] obtained a second family of quadruples (M, g, m, τ) with (0.2) or, respectively, (0.1) for any $m \geqslant 3$. This time, $\tau \neq 0$ everywhere in M. (See §44 and [5, Chapter 9, Section K].) The Kähler metric conformal to Page's metric was independently discovered by Calabi [6], [7], [9]. It follows from our classification that Page's and Bérard Bergery's examples just mentioned are the only quadruples (M, g, m, τ) with (0.2) or (0.1) which are *globally* conformally Einstein (that is, $\tau \neq 0$ everywhere in M).

More recently, Hwang and Simanca [15] and Tønnesen-Friedman [18] provided examples of (0.2) on minimal ruled surfaces M with $\tau = 0$ somewhere in M. We extend their construction by describing, in every complex dimension $m \geqslant 2$, a third family of quadruples (M, g, m, τ) that satisfy (0.1) or (0.2) and have $\tau = 0$ at some point of M. This third family is still closely related, through a form of analytic continuation, to Bérard Bergery's and Page's second family, while, for $m = 2$, it consists precisely of the examples already given in [15] and [18]. See §45.

Finally, in every odd complex dimension $m \geqslant 9$, we exhibit in §46 a new, fourth family of quadruples with (0.1). It is distinguished by a natural notion of *duality* (Remark 28.4): every quadruple in the first three families is its own dual, but none in the fourth family is.

The *moduli curve* mentioned in the title plays a prominent role in our construction. It is a subset $\mathcal{C}$ of an algebraic curve in $\mathbf{R}^2$, depending also on the complex dimension $m \geqslant 2$. Any quadruple (M, g, m, τ) with (0.1) or (0.2) gives rise to a point $(u, v) \in \mathcal{C}$, defined as follows. If g is locally reducible as a Kähler metric, $(u, v) = (0, 0)$. Otherwise, we set $u = \min \tau/c$ and $v = \max \tau/c$, with $c \in \mathbf{R} \smallsetminus \{0\}$ characterized by the property that $|\nabla\tau|^2$ is a rational function of τ having a unique real pole at c (Remark 3.2). In this way one obtains not all $(u, v) \in \mathcal{C}$, but only a countably infinite set of points that we call p*-rational*. See Remark 39.1.

The quadruple (M, g, m, τ) can in turn be explicitly reconstructed (see §1) from the corresponding p-rational point (u, v) coupled with some additional data which include a compact Kähler-Einstein manifold (N, h) with $\dim_{\mathbf{C}} N = m - 1$ such that M is a holomorphic $\mathbf{C}\mathrm{P}^1$ bundle over N. The present paper provides a proof of this fact and a relatively detailed description of the set of p-rational points; our four families arise when one requires (u, v) to lie in one of four specific subsets of $\mathcal{C}$ (see Remark 1.4). The number of p-rational points in each of the four subsets is *one* for the first family (§43) and *infinite* for the third family (§45 and Theorem 1.6). For the second and fourth families this number is *finite* and varies with m so that, as $m \to \infty$, it is asymptotic to a positive constant times m^2 (Theorem 1.5).

Although only p-rational points are directly used in our classification of (0.1) and (0.2), the other points of $\mathcal{C}$ have a similar geometric interpretation. Namely, the last two paragraphs are valid even if one replaces p-rational points with arbitrary points of $\mathcal{C}$, provided that, instead of compactness of M and N, one only requires M to be *vertically compact*, which amounts to compactness of the $\mathbf{C}\mathrm{P}^1$ fibres, but not necessarily of the base N. See §49.

1. Statement of the main results

In this section m is an integer with $m \geqslant 2$ and u, v are the Cartesian coordinates in $\mathbf{R}^2$. Most objects discussed here depend on the choice of m.

We denote by $\mathcal{H}$ the hyperbola $uv = u + v$ in $\mathbf{R}^2$ and let $\mathcal{T} \subset \mathbf{R}^2$ be the set given by $T(u, v) = 0$, where T is the symmetric polynomial of degree $3(m - 2)$ described in Lemmas 17.3 and 30.4. Thus, T and $\mathcal{T}$ depend on m. See Fig. 1.

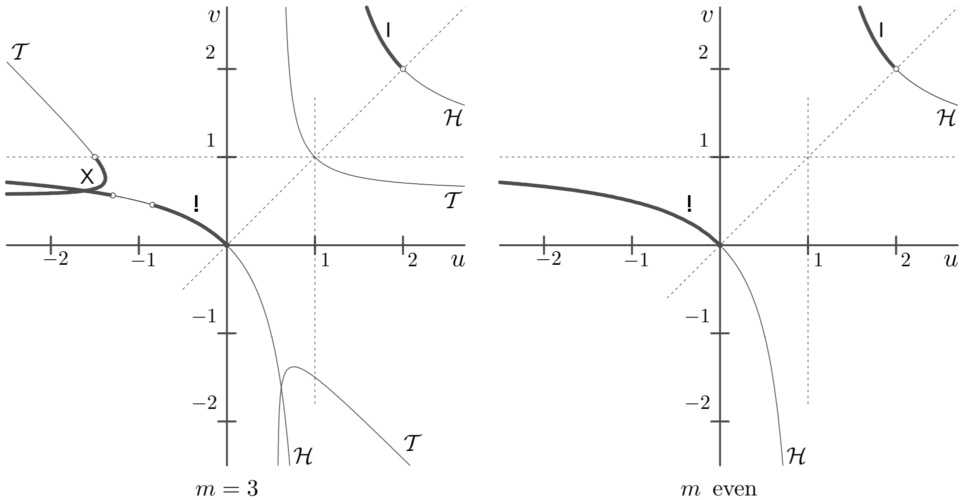

Figure 1: The components $\mathsf{!}$, I of $\mathcal{C}$ and, for odd m, the two beams of X, are the heavily marked curve segments, each contained in $\mathcal{H}$ except for the $\mathcal{T}$-beam of X, contained in $\mathcal{T}$. For $m=3$, the set $\mathcal{T}$ consists of the isolated point $(0,0)$ and three disjoint real-analytic curves in $\mathbf{R}^2$, diffeomorphic to $\mathbf{R}$. (See Remark 30.5.)

The *moduli curve* corresponding to a given value of m is a subset $\mathcal{C}$ of the half-plane $u \leqslant v$ in $\mathbf{R}^2$ such that $\mathcal{C} \subset \mathcal{H} \cup \mathcal{T}$ for odd m, while, if m is even, $\mathcal{C} = \{(u,v) \in \mathcal{H} : (u,v) \neq (1,1) \text{ and } u \leqslant v\}$. Thus, $\mathcal{C}$ is the same for all even m. The following explicit description of $\mathcal{C}$, although different from the definition given at the end of §2, is equivalent to it (see Theorem 33.1). Namely, $\mathcal{C}$ is the disjoint union of its connected components (defined below):

$$\mathcal{C} = \mathsf{!} \cup \mathsf{I} \quad \text{if } m \text{ is even}, \quad \text{and} \quad \mathcal{C} = \mathsf{X} \cup \mathsf{!} \cup \mathsf{I} \quad \text{if } m \text{ is odd}. \tag{1.1}$$

Specifically, the X component exists for odd m only and is contained in $\mathcal{H} \cup \mathcal{T}$, so that it is the union of its $\mathcal{H}$*-beam* $\mathsf{X} \cap \mathcal{H}$ and $\mathcal{T}$*-beam* $\mathsf{X} \cap \mathcal{T}$. The first two of the sets X, $\mathsf{!}$ and I in $\mathbf{R}^2$ depend on m, and the $\mathcal{T}$-beam $\mathsf{X} \cap \mathcal{T}$ of X is the intersection of $\mathcal{T}$ with $(-\infty,0) \times (0,1)$. Next, $\mathsf{X} \cap \mathcal{H}$, $\mathsf{!}$ and I are subsets of the hyperbola $\mathcal{H}$, namely, the segments of $\mathcal{H}$ that project onto the following intervals in the u axis: $(-\infty,z)$ (for $\mathsf{X} \cap \mathcal{H}$ and odd m), or $(w,0]$ (for $\mathsf{!}$ and odd m), or $(-\infty,0]$ (for $\mathsf{!}$ and even m), or $(1,2)$ (for I and any m), with constants z,w such that $z<w<0$, defined in §21, and depending on the odd integer m.

In §21 the symbol z is assigned a meaning also when m is even; the constant $z \in (-\infty,0)$ depending on an *even* integer m appears in the following definition of a function $\delta : \mathcal{C} \to \{-1,0,1\}$, which also depends on m. Namely, we set $\delta = 1$ both on X for odd m and on I for all m, as well as $\delta = -1$ on $\mathsf{!}$ for odd m and $\delta = \operatorname{sgn}(z-u)$ at any $(u,v) \in \mathsf{!}$ for even m. Thus, when m is even, $\mathsf{!}$ contains a distinguished point (z,z^*) with $z^* = z/(z-1)$, at which $\delta = 0$.

The symbols X, $\mathsf{!}$ and I imitate the topology of the sets in question: I, $\mathsf{X} \cap \mathcal{H}$, $\mathsf{X} \cap \mathcal{T}$ and $\mathsf{!}$ are real-analytic submanifolds of $\mathbf{R}^2$ and $\mathsf{!}$ is diffeomorphic to $(0,1]$, while the other three are diffeomorphic to $\mathbf{R}$, and the two beams $\mathsf{X} \cap \mathcal{H}$ and $\mathsf{X} \cap \mathcal{T}$ forming X have a single transverse intersection point. See Proposition 29.1(ii) and Remark 33.3.

We let p stand for a specific rational function of the variables u,v, depending on m, which is defined in §34. The restriction of p to $\mathcal{C}$ is finite (that is, well-defined) and nonzero everywhere in $\mathcal{C} \smallsetminus \{(0,0)\}$ (for odd m) or in $\mathcal{C} \smallsetminus \{(0,0),(z,z^*)\}$ (for even m). In addition, $p=0$ at $(0,0)$ for any m, while, for even m only, p is undefined at (z,z^*).

Definition 1.1. Given a fixed integer $m \geqslant 2$, by a *p-rational point* we mean any $(u,v) \in \mathcal{C}$ at which either $\delta = 0$ (that is, m is even and $(u,v) = (z,z^*)$), or $\delta = -1$ and the value of p is

rational, or, finally, $\delta = 1$ and p equals n/d for some $n \in \mathbf{Z}$ and $d \in \{1, \dots, m\}$. In particular, $(0,0) \in \mathbf{!} \subset \mathcal{C}$ is p-rational.

We will now describe how p-rational points on $\mathcal{C}$ are related, for any given $m \geqslant 2$, to quadruples (M, g, m, τ) with (0.1) or (0.2). A similar geometric interpretation is valid for *arbitrary* points of $\mathcal{C}$, provided that the compactness requirement in (0.1) or (0.2) is suitably relaxed. (See §49.)

Every p-rational point $(u, v) \in \mathcal{C}$, for any fixed integer $m \geqslant 2$, corresponds to a mapping which assigns a quadruple (M, g, m, τ) with (0.1) or (0.2) to any suitable input data. Those data consist of a compact Kähler-Einstein manifold (N, h) with $\dim_{\mathbf{C}} N = m - 1$, a holomorphic line bundle $\mathcal{L}$ over N, a $\mathrm{U}(1)$ connection in $\mathcal{L}$ with a curvature form Ω, and a constant $c \in \mathbf{R} \smallsetminus \{0\}$, which satisfy the following additional requirements, depending on (u, v), and involving the Ricci form $\rho^{(h)}$ of h, its Kähler form $\omega^{(h)}$, and the signum of the Einstein constant κ of (N, h). Namely, $\operatorname{sgn} \kappa$ must equal the value of $\delta : \mathcal{C} \to \{-1, 0, 1\}$ at (u, v), and, for the rational function p mentioned above, one of the following two cases has to occur:

$$\begin{array}{rl} \text{i)} & \kappa \neq 0 \text{ and } \Omega \text{ equals } \rho^{(h)} \text{ times the value of } p \text{ at } (u, v), \\ \text{ii)} & \kappa = 0 \text{ and } \Omega \text{ is a nonzero real multiple of } \omega^{(h)}. \end{array} \tag{1.2}$$

Note that $\rho^{(h)}$ and Ω, divided by 2π, represent the real Chern classes $c_1(N)$ and $c_1(\mathcal{L})$ in $H^2(N, \mathbf{R})$. Also, $\rho^{(h)} = \kappa \omega^{(h)}$. Thus, in case (i), $c_1(N) \neq 0$ and $c_1(\mathcal{L})$ equals $c_1(N)$ times the value of p at (u, v). Similarly, in case (ii), $c_1(N) = 0$ and $c_1(\mathcal{L})$ is a nonzero real multiple of the Kähler class of h. Moreover, $\Omega = 0$ in (i) only if $(u, v) = (0,0)$, while (ii) occurs only when m is even and $(u, v) = (z, z^*)$.

The quadruple (M, g, m, τ) is constructed out of such data as follows. If $(u, v) = (0,0)$, we proceed as described in §43. Now let $(u, v) \in \mathcal{C} \smallsetminus \{(0,0)\}$. By Remark 19.3, $1 < u < v$ or $u < v < 1$, that is, $I = [u, v]$ is a nontrivial closed interval and $1 \notin I$. Hence we may choose $\varepsilon \in \{1, -1\}$ with $\varepsilon c(t - 1) > 0$ for every t in the interior of I, where c comes from our data. In terms of κ and the value of p appearing in (1.2), we define $a \in \mathbf{R}$ either by $a = -\varepsilon p \kappa / 2$ (case (1.2.i)), or by $\Omega = -2\varepsilon a \omega^{(h)}$ (case (1.2.ii)). On the other hand, the definition of $\mathcal{C}$ at the end of §2 guarantees the existence of a rational function Q satisfying some specific conditions that involve I. Such Q, unique up to a constant factor (Lemma 9.1), is now made unique by requiring (3.1) to hold for Q and c, ε, a. Our data and $\varepsilon, a, p, \kappa$ also satisfy (3.2), as one sees using Remark 34.1 and noting that, as $\rho^{(h)} = \kappa \omega^{(h)}$, our choice of a gives $\Omega = -2\varepsilon a \omega^{(h)}$ in *both* cases (1.2.i), (1.2.ii).

Applying the construction described in §3 we now obtain the required quadruple (M, g, m, τ). In particular, M is the projective compactification of the total space of $\mathcal{L}$, that is, the bundle of Riemann spheres associated with $\mathcal{L}$.

Proposition 1.2. *For any p-rational point on the moduli curve $\mathcal{C}$, with any integer $m \geqslant 2$, additional data as above exist, and, applying to them the construction just described, we always obtain a quadruple (M, g, m, τ) with* (0.1) *or* (0.2).

Namely, the data exist according to Remark 39.1, while (0.1) or (0.2) is verified in §3. On the other hand, in §39 we prove the following result:

Theorem 1.3. *Conversely, every quadruple (M, g, m, τ) with* (0.1) *or* (0.2) *is, up to a τ-preserving biholomorphic isometry, obtained from the above construction applied to some p-rational point $(u, v) \in \mathcal{C}$ and some additional data with the properties just listed.*

Remark 1.4. The four families of quadruples with (0.1) or (0.2), mentioned in §0, correspond to a decomposition of the moduli curve $\mathcal{C}$ into the following four disjoint subsets, from which the p-rational points then are chosen. For the first family, it is the subset consisting of $(0,0)$ alone; for

the second, the I component; for the third, $\mathsf{!} \smallsetminus \{(0,0)\}$, augmented (only when m is odd) by the $\mathcal{H}$-beam $\mathsf{X} \cap \mathcal{H}$ of the X component; and, for the fourth family, the empty set if m is even, or, if m is odd, the $\mathcal{T}$-beam $\mathsf{X} \cap \mathcal{T}$ minus its unique intersection point with the $\mathcal{H}$-beam.

Proposition 1.2 and Theorem 1.3 together form a classification result for (0.1) and (0.2). Its meaning, however, remains obscure unless one addresses the questions of abundance or scarcity, including that of the very existence, of p-rational points corresponding to each of the four families. The next two results, proved in sections 44 – 46 and 38, provide information of this kind.

Theorem 1.5. *For any given integer $m \geqslant 2$, the set of p-rational points is a countably infinite subset of the moduli curve $\mathcal{C}$. Its intersections with I, $\mathsf{X} \cap \mathcal{H}$ and $\mathsf{X} \cap (\mathcal{T} \smallsetminus \mathcal{H})$, which we denote, respectively, by $\mathcal{S}_m^{\mathsf{I}}$, $\mathcal{S}_m^{\mathcal{H}}$ and $\mathcal{S}_m^{\mathcal{T}}$, are all finite, and their cardinalities satisfy the asymptotic relations*

$$\begin{array}{llll} \text{a)} & |\mathcal{S}_m^{\mathsf{I}}|/m^2 \;\to\; 3/\pi^2 & \text{as} & m \to \infty, \\ \text{b)} & |\mathcal{S}_m^{\mathcal{H}}|/m^2 \;\to\; 9\sqrt{2}(\sqrt{2}+1)/\pi^2 & \text{as} & (\text{odd})\ m \to \infty, \\ \text{c)} & |\mathcal{S}_m^{\mathcal{T}}|/m^2 \;\to\; (6-4\sqrt{2})/\pi^2 & \text{as} & (\text{odd})\ m \to \infty, \end{array}$$

where X, $\mathcal{S}_m^{\mathcal{H}}$ and $\mathcal{S}_m^{\mathcal{T}}$ are defined for odd m only. The values assumed by p on $\mathcal{S}_m^{\mathcal{H}}$, or $\mathcal{S}_m^{\mathcal{T}}$, all lie in the interval $(-2,0)$, or, respectively, $(-1,0)$. In addition,

- d) *p maps $\mathcal{S}_m^{\mathsf{I}}$ bijectively onto the set of all rational numbers in $(0,1)$ with positive denominators not exceeding m, cf.* Remark 42.2.
- e) *If m is odd, p assumes the values -1 and $-1+1/m$ at a unique pair of points of $\mathcal{S}_m^{\mathcal{H}}$. Therefore, $|\mathcal{S}_m^{\mathcal{H}}| \geqslant 2$.*
- f) *$\mathcal{S}_m^{\mathcal{T}} = \emptyset$ if $m \in \{3,5,7\}$, while $|\mathcal{S}_m^{\mathcal{T}}| \geqslant 2$ if m is odd and $m \geqslant 9$. In particular, $|\mathcal{S}_9^{\mathcal{T}}| = 2$ and $|\mathcal{S}_m^{\mathcal{T}}| \geqslant 2(m-17)$ for any odd $m \geqslant 19$.*

The limits in (a) – (c) are, approximately, 0.304, 3.113 and 0.035. For the remaining component $\mathsf{!}$, we have the following theorem.

Theorem 1.6. *Given an integer $m \geqslant 2$, the set of p-rational points in the $\mathsf{!}$ component of $\mathcal{C}$ is countably infinite.*

- i) *If m is odd, the values of p on $\mathsf{!}$ all lie in $(0, 1/m)$.*
- ii) *For even m, the set of those p-rational points in by p bijectively onto the set of rational numbers in $(-\infty,-1)$ that have positive denominators not exceeding m or, respectively, onto the set of rational numbers in $[0,\infty)$.*

2. The moduli curve

Following [10] we set, for any fixed integer $m \geqslant 1$,

$$F(t) \;=\; \frac{(t-2)t^{2m-1}}{(t-1)^m}\,, \qquad E(t) \;=\; (t-1)\sum_{j=1}^{m} \frac{j}{m}\binom{2m-j-1}{m-1} t^{j-1}. \tag{2.1}$$

Thus, F and E are rational functions of the real variable t and

$$E(t) \;=\; (t-1)\Sigma(t)\,, \qquad \text{where} \qquad \Sigma(t) \;=\; \sum_{j=1}^{m} \frac{j}{m}\binom{2m-j-1}{m-1} t^{j-1}. \tag{2.2}$$

Any real constants A, B, C now give rise to a rational function Q with

$$Q(t) \;=\; (t-1)\,[A \,+\, BE(t) \,+\, CF(t)]\,, \qquad \text{for } E, F \text{ as in (2.1)}. \tag{2.3}$$

(Cf. [10, formula (21.5)].) For a fixed integer $m \geqslant 2$, let

$$\mathbf{V} = \operatorname{Span}\{t-1,\,(t-1)E,\,(t-1)F\}\,, \qquad \text{with} \quad E, F \ \text{ as in (2.1)}, \tag{2.4}$$

t being the identity function. Thus, $\mathbf{V}$ is the 3-dimensional real vector space, depending on m, of all rational functions of the form (2.3) with $A, B, C \in \mathbf{R}$.

Given a nontrivial closed interval $I \subset \mathbf{R}$, we may impose on $Q \in \mathbf{V}$ and I one or more of the following five conditions (cf. [11, formula (34.2)]):

$$\begin{array}{rl} \text{a)} & Q \text{ is analytic on } I, \text{ that is, } I \text{ does not contain } 1 \text{ unless } C = 0 \text{ in (2.3).} \\ \text{b)} & Q = 0 \text{ at both endpoints of } I. \\ \text{c)} & Q \neq 0 \text{ at all interior points of } I \text{ at which } Q \text{ is analytic.} \\ \text{d)} & \dot{Q} = dQ/dt \text{ exists and is nonzero at both endpoints of } I. \\ \text{e)} & \text{The values of } \dot{Q} \text{ at the endpoints of } I \text{ exist and are mutually opposite.} \end{array} \tag{2.5}$$

Remark 2.1. Conditions (2.5.a,c) alone imply that 1 cannot be an interior point of I. In fact, if $1 \in I$, (2.5.a) gives $C = 0$, and so $Q(1) = 0$ due to (2.3). Thus, by (2.5.c), 1 is an endpoint of I.

Let $m \geqslant 2$ be a fixed integer. We define the *moduli curve* to be the set $\mathcal{C} \subset \mathbf{R}^2$ consisting of $(0,0)$ and all (u, v) with $u < v$ for which there exists a function Q in the space $\mathbf{V}$ given by (2.4), satisfying all of (2.5) on the interval $I = [u, v]$.

3. The main step in the construction

The simplest examples of quadruples (M, g, m, τ) satisfying (0.1) or (0.2) are described in §43; they involve Kähler metrics g that are locally reducible. The following construction, which also appears in [11], leads to quadruples with (0.1) or (0.2) that are *not* locally reducible (cf. Remark 43.1).

Let a function $Q \in \mathbf{V}$, with $\mathbf{V}$ as in (2.4) for a fixed integer $m \geqslant 2$, satisfy all five conditions (2.5) on a nontrivial closed interval $I = [u, v]$ with $1 \notin I$. Replacing Q by $-Q$, if necessary, we also require (cf. (2.5.c)) that $Q > 0$ on the open interval (u, v). We now choose $c, \varepsilon, a \in \mathbf{R}$ with

$$\begin{array}{l} \varepsilon \in \{1, -1\} \ \text{ and } \ \varepsilon c(t-1) > 0 \ \text{ for every } t \text{ in the open interval } (u, v), \\ \text{while } \ \dot{Q}(u) = -2ac \ \text{ and } \ \dot{Q}(v) = 2ac, \ \text{ with } \dot{Q} = dQ/dt. \end{array} \tag{3.1}$$

(Such c, ε, a must exist by (2.5.a,c,e).) Next, let there be given

$$\begin{array}{l} \text{a compact Kähler-Einstein manifold } (N, h) \text{ of complex dimension } m-1 \\ \text{having the Ricci form } \rho^{(h)} = \kappa\omega^{(h)} \text{ for } \kappa = \varepsilon m A/c, \text{ a complex line bundle } \mathcal{L} \\ \text{over } N, \text{ and a } \mathrm{U}(1) \text{ connection in } \mathcal{L} \text{ with the curvature form } \Omega = -2\varepsilon a\omega^{(h)}, \end{array} \tag{3.2}$$

where A is determined by Q via (2.3) and $\omega^{(h)}$ denotes the Kähler form of (N, h). The question whether such objects exist is discussed in §4.

With $m, Q, I, c, \varepsilon, a$ and the objects (3.2) fixed as above, let us also choose a positive function r of the variable t restricted to the interior of I, such that $dr/dt = acr/Q$. By (2.5.b) – (2.5.d), $\log r$ and r have the ranges $(-\infty, \infty)$ and $(0, \infty)$. We may thus treat t along with $\tau = ct$ and Q, restricted to the interior of I, as functions of a new variable $r \in (0, \infty)$, so that $u = \inf t$ and $v = \sup t$ for $t : (0, \infty) \to \mathbf{R}$.

The total space of the line bundle in (3.2) is denoted by the same symbol $\mathcal{L}$, and r also stands for the function $\mathcal{L} \to (0, \infty)$ which, restricted to each fibre, is the norm corresponding to the $\mathrm{U}(1)$ structure. Being functions of $r > 0$, both t and $\tau = ct$, as well as Q, become functions on $\mathcal{L} \smallsetminus N$, where $N \subset \mathcal{L}$ is the zero section.

We now define a metric g on the complex manifold $\mathcal{L} \smallsetminus N$ by letting g on each fibre of $\mathcal{L}$ coincide with $Q/(ar)^2$ times the standard Euclidean metric, declaring the horizontal distribution of

the connection in $\mathcal{L}$ to be g-normal to the fibres, and requiring that g restricted to the horizontal distribution equal $2|\tau - c|$ times the pullback of h under the projection $\mathcal{L} \to N$.

Finally, let the compact complex manifold M be the projective compactification of $\mathcal{L}$, that is, the Riemann sphere bundle obtained when the total spaces of $\mathcal{L}$ and its dual $\mathcal{L}^*$ are glued together by the biholomorphism $\mathcal{L} \smallsetminus N \to \mathcal{L}^* \smallsetminus N$ which sends each $\phi \in \mathcal{L}_y \smallsetminus \{0\}$, $y \in N$, to the unique $\chi \in \mathcal{L}_y^*$ with $\chi(\phi) = 1$.

Theorem 34.3 in [11] now shows that g and $\tau = ct$ have C^∞ extensions to M such that the resulting quadruple (M, g, m, τ) satisfies (0.1) or (0.2).

This yields Proposition 1.2, except for the existence assertion, proved later in Remark 39.1. Also, the above inf/sup relations give $u = \min t$, $v = \max t$ on M.

Remark 3.1. Equation $dr/dt = acr/Q$ determines r only up to a positive constant factor. A different choice of r thus amounts to rescaling the Hermitian fibre metric in $\mathcal{L}$, and the resulting quadruple is equivalent to the original one under an obvious biholomorphic isometry.

Remark 3.2. The constants c, u, v used in the above construction are in turn uniquely determined by the biholomorphic-isometry type of the resulting quadruple (M, g, m, τ). Specifically, one easily sees that the g-gradient $\nabla\tau$ of τ equals a times the "identity" vertical field on $\mathcal{L}$, which in turn gives $g(\nabla\tau, \nabla\tau) = Q(\tau/c)$. Thus, c is the unique real pole of $|\nabla\tau|^2$ treated as a rational function of the variable τ (cf. Remark 33.4), while $u = \min \tau/c$ and $v = \max \tau/c$, since $\tau/c = t$.

A purely local definition of c is also possible, even in a much more general situation; see [10, Lemma 12.5 (and Corollary 9.3)]. Finally, if m is fixed, u and v depend only on the homothety class of the Riemannian metric on the 2-sphere obtained by restricting g to some, or any, fibre of the $\mathbf{C}\mathrm{P}^1$ bundle M. (We will not use this fact, which follows since τ is a Killing potential, while the fibre geometry determines the corresponding Killing field uniquely up to a factor.)

4. Rationality conditions

We will use a result of Kobayashi and Ochiai [16], as quoted in subsection 9.124 of [5]: if N is a compact complex manifold with $\dim_{\mathbf{C}} N = m - 1$ such that $c_1(N) \in H^2(N, \mathbf{Z})$ is positive and divisible by an integer $d \geqslant 1$, then $d \leqslant m$, with equality only if N is biholomorphic to $\mathbf{C}\mathrm{P}^{m-1}$.

For $m, Q, I, u, c, \varepsilon, A$ as in (3.1) – (3.2), let $p \in \mathbf{R}$ and $\delta \in \{-1, 0, 1\}$ be given by

$$\text{i)} \quad mp = \dot{Q}(u)/A \quad (\text{only if } A \neq 0), \qquad \text{ii)} \quad \delta = \operatorname{sgn} \kappa, \quad \text{where} \quad \kappa = \varepsilon m A/c, \tag{4.1}$$

sgn being the usual signum function with $\operatorname{sgn} 0 = 0$ and $\operatorname{sgn} \xi = \xi/|\xi|$ for $\xi \in \mathbf{R} \smallsetminus \{0\}$.

The invariant δ in (4.1.ii) depends just on the original m, Q, I, and not on c or ε. Both p, defined only if $A \neq 0$, and δ remain unaffected when Q is multiplied by a positive constant. In fact, I and Q determine $\operatorname{sgn}(\varepsilon c)$ via (a), while rescaling Q leads to multiplication of A in (2.3) and $\dot{Q}$ by the same positive factor.

Unlike c, ε, a in (3.1), the objects (3.2) need not exist. Moreover, whether they exist or not depends just on m, Q, I, and not on how we chose c, ε, a. Namely, let p, δ be determined by m, Q and I as in (4.1). Then (3.2) holds for some $N, h, \mathcal{L}$ and a $\mathrm{U}(1)$ connection in $\mathcal{L}$, if and only if

$$\begin{array}{l} \text{either } \delta = 1 \text{ and } p = n/d \text{ for some } n \in \mathbf{Z} \text{ and } d \in \{1, \dots, m\}, \\ \text{or } \delta = 0, \quad \text{or,} \quad \text{finally,} \quad \delta = -1 \text{ and } p \text{ is rational.} \end{array} \tag{4.2}$$

In fact, given (3.2) with $\kappa = A = 0$, we have $\delta = 0$ by (4.1.ii), and (4.2) follows. Also, if $\kappa \neq 0$ in (3.2) (so that $A \neq 0$), then (4.1) and the Kobayashi-Ochiai theorem mentioned above give (4.2) with $\delta = \pm 1$ (cf. Remark 4.1 below).

Conversely, let (4.2) (and (3.1)) be satisfied. If $A = 0$, (3.2) is easily realized by choosing (N, h) to be a compact Ricci-flat Kähler manifold whose Kähler class equals $-\varepsilon\pi/a$ times an integral class (for instance, a suitable flat complex torus). We then select $\mathcal{L}$ so that the latter class is $c_1(\mathcal{L})$, and hence $\varOmega = -2\varepsilon a\omega^{(h)}$ is the curvature form of some $\mathrm{U}(1)$ connection in $\mathcal{L}$.

Finally, let us assume (4.2) with $A \neq 0$, that is, $\delta = \pm 1$. Thus, $p = n/d$ with relatively prime integers n and $d \geqslant 1$. For any integer $s \geqslant 1$, let N be the *Fermat hypersurface* of degree s in $\mathbf{C}\mathrm{P}^m$, given by $z_0^s + z_1^s + \ldots + z_m^s = 0$ in homogeneous coordinates $z_0, \ldots, z_m$. The adjunction formula [12, p. 147] implies that $c_1(N) = (m + 1 - s)[e]$ in $H^2(N, \mathbf{R})$, where $[e]$ is the restriction to N of the positive generator of $H^2(\mathbf{C}\mathrm{P}^m, \mathbf{R})$. This has three consequences. First, if $s > m+1$, or $s \in \{1, \ldots, m\}$, then $c_1(N) < 0$ or, respectively, $c_1(N) > 0$, and so a Kähler-Einstein metric h on N exists in view of the Aubin and Yau solution to Calabi's conjecture [2], [19] or, respectively, by a recent result of Tian [17]. Next, if $s = m + 1 - \delta d \geqslant 1$, then $\kappa, \delta, m + 1 - s, c_1(N)$ and the Ricci form $\rho^{(h)}$ of h all have the same sign; hence, rescaling h, we can always ensure that $\rho^{(h)} = \kappa\omega^{(h)}$. Third, if $\mathcal{E}$ is the restriction to N of the dual of the tautological bundle over $\mathbf{C}\mathrm{P}^m$ and $\mathcal{L} = \mathcal{E}^{\otimes n}$, then $c_1(\mathcal{L}) = pc_1(N)$ in $H^2(N, \mathbf{R})$, as both sides equal $n[e]$. Now (3.2) follows, with a connection in $\mathcal{L}$ chosen as in Remark 4.2 below.

Remark 4.1. We clearly have $\varOmega = p\rho^{(h)}$ and $c_1(\mathcal{L}) = pc_1(N)$ in $H^2(N, \mathbf{R})$, for p as in (4.1), whenever (3.1) – (3.2) hold with $A \neq 0$. Since both Chern classes are integral, p must be a rational number and its irreducible denominator divides $c_1(N)$ in $H^2(N, \mathbf{Z})$.

Remark 4.2. If $\mathcal{L}$ is a complex line bundle over a compact Kähler manifold (N, h) such that $c_1(\mathcal{L}) = pc_1(N)$ in $H^2(N, \mathbf{R})$ for some (necessarily rational) number p, then $\varOmega = p\rho^{(h)}$ is the curvature form of some $\mathrm{U}(1)$ connection in $\mathcal{L}$. In fact, $2\pi\varOmega$ represents $c_1(\mathcal{L})$ in $H^2(N, \mathbf{R})$, since $\rho^{(h)}$ represents $c_1(N)$.

Remark 4.3. The importance of the objects $N, h, \mathcal{L}$ in (3.2) is due to their role as building blocks for the construction in §3, leading to quadruples (M, g, m, τ) with (0.1) or (0.2). As the case where such $N, h, \mathcal{L}$ exist is completely characterized by (4.2), the next natural question concerns the extent of freedom in choosing $N, h, \mathcal{L}$, for any fixed m, Q, I having the properties listed immediately before (3.1), along with (4.2). When $A \neq 0$ in (2.3), we can make the following comments.

In view of Remark 4.1 and [11, Remark 2.4], once the Kähler manifold (N, h) is selected, the choices of $\mathcal{L}$ become quite limited: up to tensoring by holomorphic line bundles with flat $\mathrm{U}(1)$ connections, $\mathcal{L} = \mathcal{E}^{\otimes p}$ for the anticanonical bundle $\mathcal{E} = [TN]^{\wedge(m-1)}$ of N. (A connection required in (3.2) exists by Remark 4.2.)

As for selecting (N, h), there are two interesting special cases. First, (4.2) obviously holds if the rational number p (cf. Remark 4.1) corresponding to the given m, Q, I with $A \neq 0$ is an integer; our (N, h) then can be *any* compact Kähler-Einstein manifold (N, h) with $\dim_{\mathbf{C}} N = m - 1$ that has the correct value of the Einstein constant κ, with $\mathcal{L}$ as in the last paragraph.

An opposite extreme occurs, when (4.1), for our m, Q, I with $A \neq 0$, gives $\delta = 1$ and $p = n/m$ for an integer n such that n, m are relatively prime (which clearly implies (4.2)). The objects realizing (3.2) then are *essentially unique*: (N, h) must be biholomorphically isometric to $\mathbf{C}\mathrm{P}^{m-1}$ with a constant multiple of the Fubini-Study metric, in such a way that $\mathcal{L}$ becomes the nth tensor power of the dual tautological bundle. This is due to the equality clause in the Kobayashi-Ochiai theorem (see the beginning of this section), since a holomorphic line bundle $\mathcal{L}$ over $N = \mathbf{C}\mathrm{P}^{m-1}$ is uniquely determined by $c_1(\mathcal{L}) \in H^2(N, \mathbf{R})$, while a Kähler-Einstein metric on $\mathbf{C}\mathrm{P}^{m-1}$ is essentially unique ([12, pp. 144–145], and [3]).

5. Some functional relations

Throughout this section, F, E and Σ are the functions with (2.1) – (2.2) for a fixed integer $m \geqslant 2$. By (2.1), the derivative $\dot{F} = dF/dt$ is given by

$$\text{a)}\quad \dot{F}(t) \,=\, \frac{t^{2m-2}}{(t-1)^{m+1}}\,\Lambda(t)\,, \qquad \text{where} \quad \text{b)}\quad \Lambda(t) \,=\, mt^2 \,-\, 2(2m-1)(t-1) \,>\, 0. \tag{5.1}$$

The dependence of F, E, Σ on m will usually be suppressed in our notation. Right now, however, we make it explicit by writing F_m, E_m, Σ_m rather than F, E, Σ. For $m \geqslant 2$ one then has, as in [10, the paragraph preceding (21.3)],

$$\text{i)}\quad F_m(t) \,=\, \frac{t^2}{t-1}\,F_{m-1}(t), \quad \text{ii)}\quad E_m(t) \,=\, \frac{t^2}{t-1}\,E_{m-1}(t) \,-\, \frac{1}{m}\binom{2m-2}{m-1}, \quad E_1(t) \,=\, t-1\,. \tag{5.2}$$

Here (5.2.i) follows from (2.1) and (5.2.ii) is obtained by expanding the difference of the two sides into powers of t via (2.1) – (2.2). Thus,

$$\frac{E}{F} \,=\, \frac{E_{m-1}}{F_{m-1}} \,-\, \binom{2m-2}{m-1}\frac{1}{mF}\,, \qquad \text{where} \quad E = E_m \ \text{and} \ F = F_m. \tag{5.3}$$

Consequently, for every $t \in \mathbf{R} \smallsetminus \{0,1,2\}$, induction on $m \geqslant 2$ gives

$$\frac{d}{dt}\,[E/F] \,=\, -\binom{2m}{m}\frac{(t-1)^m}{(t-2)^2 t^{2m}}\,. \tag{5.4}$$

Specifically, the inductive step comes from (5.3), where one differentiates $1/F$ using (5.1.a), then replaces F with the expression in (2.1), and uses (5.1.b). Thus,

$$\begin{array}{rl} \text{i)} & t(t-1)(t-2)\dot{F}(t) \,=\, \Lambda(t)F(t) \qquad \text{for all } t \in \mathbf{R} \smallsetminus \{1\}, \\ \text{ii)} & t(t-1)(t-2)\dot{E}(t) \,=\, \Lambda(t)E(t) \,-\, 2(2m-1)(t-1)\Sigma(0) \ \text{ for } t \in \mathbf{R}, \end{array} \tag{5.5}$$

for Λ as in (5.1.b). Namely, (5.5.i) is obvious from (5.1.a) and (2.1), while (5.5.ii) follows if one rewrites (5.4) multiplied by F using the quotient rule for derivatives, (5.5.i), and the definitions of F, Σ in (2.1), (2.2). Also, by (2.1) – (2.2),

$$\text{i)}\quad \Sigma(0) \,=\, \dot{\Sigma}(0) \,=\, \frac{1}{m}\binom{2m-2}{m-1}, \quad \text{ii)}\quad E(0) \,=\, -\,\Sigma(0)\,, \quad \text{iii)}\quad \dot{E}(0) \,=\, 0\,. \tag{5.6}$$

Remark 5.1. The coefficients of the polynomial Σ given by (2.2), for any fixed integer $m \geqslant 1$, are all positive, and so $\Sigma(t) > 0$ whenever $t \geqslant 0$. Therefore, all real roots of Σ are negative. The same applies to the derivative $\dot{\Sigma}$ when $m \geqslant 2$. Thus, by (2.2), $E(u) < 0 < E(v)$ whenever $0 \leqslant u < 1 < v$.

If $Q \in \mathbf{V}$, with $\mathbf{V}$ as in (2.4) for a fixed integer $m \geqslant 2$, and $A, B, C \in \mathbf{R}$ represent Q in (2.3), then $t(t-1)(t-2)\dot{Q}(t) = [t(t-2) + \Lambda(t)]Q(t) - (t-1)A\Lambda(t) - 2(2m-1)(t-1)^2 B\Sigma(0)$, as one sees using (2.3), (5.5) and, again, (2.3). Thus, $u(u-2)\dot{Q}(u) = -A\Lambda(u) - 2(2m-1)(u-1)B\Sigma(0)$ whenever $u \in \mathbf{R} \smallsetminus \{1\}$ and $Q(u) = 0$. Hence, if $A \neq 0$, (5.1.b) yields, for p as in (4.1.i),

$$p \,=\, \frac{2(u-1)\lambda - u^2}{(u-2)u} \quad \text{whenever } u \in \mathbf{R} \smallsetminus \{0,1,2\} \text{ and } Q(u) = 0, \tag{5.7}$$

where, for any given $u \in \mathbf{R} \smallsetminus \{1\}$ with $Q(u) = 0$, we define $\lambda \in \mathbf{R}$ by

$$\lambda \,=\, (2 \,-\, 1/m)(1 \,-\, B\Sigma(0)/A)\,. \tag{5.8}$$

6. Monotonicity intervals

Let F, E be defined by (2.1) with an integer $m \geqslant 2$. By (2.1) – (2.2), (5.1) and (5.4), F (or, E/F) has a nonzero derivative everywhere in $\mathbf{R} \smallsetminus \{0, 1\}$ (or, respectively, in $\mathbf{R} \smallsetminus \{0, 1, 2\}$). One also easily sees that the rational functions F and E/F of the real variable t have the values/limits at $\pm\infty, 0, 1, 2$ listed below. One-sided limits, if different, are separated by vertical arrows indicating the direction of the jumps. The slanted arrows show which kind of monotonicity the given function has on each of the four intervals forming $\mathbf{R} \smallsetminus \{0, 1, 2\}$.

value or limit at:	$-\infty$		0		1		2		$+\infty$
for F (m even):	$+\infty$	$\searrow$	0	$\searrow$	$-\infty$	$\nearrow$	0	$\nearrow$	$+\infty$
for F (m odd):	$-\infty$	$\nearrow$	0	$\nearrow$	$+\infty\!\downarrow\!-\infty$	$\nearrow$	0	$\nearrow$	$+\infty$
for E/F (m even):	1	$\searrow$	$-\infty\!\uparrow\!+\infty$	$\searrow$	0	$\searrow$	$-\infty\!\uparrow\!+\infty$	$\searrow$	1
for E/F (m odd):	1	$\nearrow$	$+\infty\!\downarrow\!-\infty$	$\nearrow$	0	$\searrow$	$-\infty\!\uparrow\!+\infty$	$\searrow$	1

(6.1)

7. Some inequalities

For any fixed integer $m \geqslant 2$ we have, with F, E as in (2.1),

$$\begin{array}{l} \text{a)}\quad E < F \text{ on } (-\infty, 0], \quad \text{b)}\ E > 0 \text{ on } (1, \infty), \quad \text{c)}\ E > F \text{ on } (1, \infty), \\ \text{d)}\quad \text{If } m \text{ is odd, } E < 0 \text{ on } (-\infty, 1). \end{array} \tag{7.1}$$

In fact, (7.1.b) is clear from Remark 5.1. Next, (7.1.a) and (7.1.c) follow since (6.1) gives $E/F < 1$ and $F > 0$ on $(-\infty, 0)$ for even m, and $E/F > 1$ and $F < 0$ on $(-\infty, 0)$ for odd m, while, for all m, it yields $E/F < 0$ and $F < 0$ on $(1, 2)$ (so that $F < 0 < E$ there), as well as $E/F > 1$ and $F > 0$ on $(2, \infty)$. (Also, $E(0) < 0 < E(2)$ and $F(0) = F(2) = 0$ by Remark 5.1 and (2.1).) Finally, for odd m, we have $E < 0$ both on $(-\infty, 0]$ (from the inequalities just listed) and on $(0, 1)$ (from Remark 5.1), which implies (7.1.d).

8. A linear-independence property

With $E(t)$ as in (2.1) for a fixed integer $m \geqslant 2$, we define a polynomial function $\mathbf{r} : \mathbf{R} \to \mathbf{R}^3$ by

$$\mathbf{r}(t) \,=\, (t-1)^m \mathbf{i} \,+\, (t-1)^m E(t)\mathbf{j} \,+\, (t-2)t^{2m-1}\mathbf{k}\,, \tag{8.1}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form the standard basis of $\mathbf{R}^3$. Note that, as $\dot{E}(0) = 0$ by (5.6.iii),

$$\mathbf{r}(0) \,=\, -\,\dot{\mathbf{r}}(0)/m \,=\, (-1)^m\,[\,\mathbf{i} \,+\, E(0)\mathbf{j}\,]\,, \qquad \mathbf{r}(1) \,=\, -\,\mathbf{k}\,. \tag{8.2}$$

Proposition 8.1. *For any integer $m \geqslant 2$ and $u, v \in \mathbf{R}$ with $u \neq v$, the vectors $\mathbf{r}(u), \mathbf{r}(v)$ defined as in* (8.1) *are linearly independent.*

Proof. Writing $\mathbf{r} = (\xi, \eta, \zeta)$ we have $\xi(t) \neq 0$ for $t \neq 1$, while $\mathbf{r}(1) = (0, 0, -1)$ by (8.2). Therefore, our assertion follows if one of u, v equals 1. On the other hand, $\zeta(t)/\xi(t) = F(t)$ for $t \neq 1$, with $F(t)$ as in (2.1). This proves our assertion in the case where u and v are both greater than 1 or both less than 1 since, according to (6.1), F is injective both on $(-\infty, 1)$ and on $(1, \infty)$.

Therefore, switching u and v if necessary, we may assume that $u < 1 < v$. Contrary to our assertion, let $\mathbf{r}(u)$ and $\mathbf{r}(v)$ be linearly *dependent*. Now (8.1) gives $(u-1)^{-m}\mathbf{r}(u) = (v-1)^{-m}\mathbf{r}(v)$, and hence, by (2.1), $F(u) = F(v)$ and $E(u) = E(v)$. However, the last equality contradicts the relation $E(u) < E(v)$, which is immediate both when $0 \leqslant u < 1 < v$ (see Remark 5.1) and in the case where $u < 0 < 1 < v$ (since (7.1.a,c) then yield $E(u) < F(u) = F(v) < E(v)$). This contradiction completes the proof. $\square$

9. Condition (2.5.b) alone

For $Q \in \mathbf{V}$ for $\mathbf{V}$ as in (2.4) with a fixed integer $m \geqslant 2$, we have

$$\begin{array}{rl} \text{i)} & (t-1)^{m-1}Q(t) \,=\, \mathbf{p}\cdot\mathbf{r}(t)\,, \quad \text{and} \\ \text{ii)} & (t-1)^m\dot{Q}(t) \,=\, (t-1)\,\mathbf{p}\cdot\dot{\mathbf{r}}(t) \,-\, (m-1)\,\mathbf{p}\cdot\mathbf{r}(t)\,, \quad \text{with} \\ \text{iii)} & \mathbf{p} \,=\, A\mathbf{i} \,+\, B\mathbf{j} \,+\, C\mathbf{k} \in \mathbf{R}^3, \end{array} \tag{9.1}$$

where $\mathbf{r}(t)$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are as in (8.1), while $\cdot$ denotes the inner product of $\mathbf{R}^3$, and A, B, C correspond to Q via (2.3). (In fact, (2.3) gives (9.1.i), and differentiation then leads to (9.1.ii).) Thus, given such $Q, \mathbf{V}$ and m, we have

$$\begin{array}{rl} \text{a)} & \dot{Q}(0) \,=\, 0 \quad \text{whenever} \quad Q \in \mathbf{V} \quad \text{and} \quad Q(0) \,=\, 0, \\ \text{b)} & \text{If } \ Q \in \mathbf{V}, \text{ then } \ Q(u) \,=\, 0 \ \text{ if and only if } \ \mathbf{p}\cdot\mathbf{r}(u) \,=\, 0, \end{array} \tag{9.2}$$

for any $u \in \mathbf{R}$, where $\mathbf{p} \in \mathbf{R}^3$ corresponds to Q as in (9.1.iii). Namely, from (8.2) we have $\mathbf{p}\cdot\mathbf{r}(0) = -\,\mathbf{p}\cdot\dot{\mathbf{r}}(0)/m = (-1)^m\,[A + BE(0)]$ which, combined with (9.1), gives $\dot{Q}(0) = -Q(0) = A + BE(0)$, and hence (9.2.a). Next, when $u \neq 1$, (9.2.b) is clear from (9.1.i), while, if $u = 1$, condition $Q(1) = 0$ is, by (2.3) with (2.1), equivalent to $C = 0$, that is (cf. (9.1.iii) and (8.2)), to $0 = \mathbf{p}\cdot\mathbf{k} \,=\, -\,\mathbf{p}\cdot\mathbf{r}(1)$.

Lemma 9.1. *Given an integer $m \geqslant 2$ and a nontrivial closed interval $I \subset \mathbf{R}$, those $Q \in \mathbf{V}$ which vanish at both endpoints of I form a one-dimensional vector subspace of the space $\mathbf{V}$ with* (2.4). *Thus, $Q \in \mathbf{V} \smallsetminus \{0\}$ satisfying* (2.5.b) *on I exists, for any such m, I, and is unique up to a nonzero constant factor.*

Explicitly, up to a factor, $(t-1)^{m-1}Q(t) = [\mathbf{r}(u)\times\mathbf{r}(v)]\cdot\mathbf{r}(t)$, where u, v denote the endpoints of I and $\mathbf{r}$ is given by (8.1), *while $\cdot$ and $\times$ denote the inner product and vector product in $\mathbf{R}^3$.*

In fact, up to a factor, $\mathbf{p}$ in (9.1.i) equals $\mathbf{r}(u)\times\mathbf{r}(v) \neq \mathbf{0}$, since $\mathbf{r}(u)$, $\mathbf{r}(v)$ are linearly independent (Proposition 8.1) and orthogonal to $\mathbf{p}$ (by (9.2.b)).

10. A determinant formula for $Q(t)$

Given an integer $m \geqslant 2$ and a nontrivial closed interval I, let $Q \in \mathbf{V} \smallsetminus \{0\}$, with $\mathbf{V}$ as in (2.4), be a function satisfying condition (2.5.b) on I. Such Q, which exists and is unique up to a factor, for any m and I, is explicitly described in Lemma 9.1. Let us now also assume that $u \neq 1 \neq v$, where u and v are the endpoints of I. With F, E as in (2.1) for our fixed $m \geqslant 2$, (8.1) and (2.1) give $\mathbf{r}(t) \,=\, (t-1)^m\mathbf{w}(t)$ for $t \neq 1$, with $\mathbf{w}(t) \,=\, \mathbf{i} \,+\, E(t)\mathbf{j} \,+\, F(t)\mathbf{k}$. Thus, $(t-1)^{-1}Q(t) = (u-1)^m(v-1)^m\,[\mathbf{w}(u)\times\mathbf{w}(v)]\cdot\mathbf{w}(t)$ by Lemma 9.1, that is, up to another factor,

$$Q(t) \,=\, (t-1)H(t) \qquad \text{for all} \quad t \in \mathbf{R}\smallsetminus\{1\}, \qquad \text{where} \quad H(t) \,=\, \det\begin{bmatrix} 1 & E(u) & F(u) \\ 1 & E(v) & F(v) \\ 1 & E(t) & F(t) \end{bmatrix}. \tag{10.1}$$

Writing H, F, E for $H(t)$, $F(t)$, $E(t)$, and $(\;)^\cdot = d/dt$, we have, by (10.1),

$$\begin{array}{rl} \text{i)} & H \,=\, E_0F_1 \,-\, F_0E_1 \,+\, (F_0 \,-\, F_1)E \,+\, (E_1 \,-\, E_0)F\,, \\ \text{ii)} & \dot{H} \,=\, (F_0 \,-\, F_1)\dot{E} \,+\, (E_1 \,-\, E_0)\dot{F}\,, \qquad \text{where} \\ \text{iii)} & E_0 \,=\, E(u)\,, \quad E_1 \,=\, E(v)\,, \quad F_0 \,=\, F(u)\,, \quad F_1 \,=\, F(v)\,. \end{array} \tag{10.2}$$

Lemma 10.1. *Let $\mathbf{V}$ be the space* (2.4) *for a fixed integer $m \geqslant 2$, and let a nontrivial closed interval $I \subset \mathbf{R}$ have the endpoints u, v with $u \neq 1 \neq v$. If a function $Q \in \mathbf{V}$ satisfies condition* (2.5.b) *on I, that is, $Q(u) = Q(v) = 0$, while $(A, B, C) \in \mathbf{R}^3$ corresponds to Q as in* (2.3), *then, up to a nonzero overall factor,*

$$(A, B, C) \,=\, (E(u)F(v) \,-\, F(u)E(v),\; F(u) - F(v),\; E(v) - E(u))\,. \tag{10.3}$$

This is clear from (2.3) and (10.2.i,iii), as $Q(t) = (t-1)H(t)$ by (10.1).

11. A convexity lemma

According to §6, the variable t may, on suitable intervals, be diffeomorphically replaced with F. As shown next, this makes E or $-E$ a convex function of F.

Lemma 11.1. *Let $F, E, \varLambda$ be as in* (2.1) *and* (5.1) *for an integer $m \geqslant 2$. Then*

$$\frac{d}{dt}\,[\dot E/\dot F] \,=\, \binom{2m}{m}\frac{m(t-1)^m}{[\varLambda(t)]^2\,t^{2m-2}} \quad \text{at every real} \quad t \neq 0, \ \text{ with } (\ )\dot{} = d/dt. \tag{11.1}$$

In fact, let $\gamma = \binom{2m}{m}$. Writing F for $F(t)$, etc., let us differentiate (5.4) multiplied by F^2 and then multiply the result by $\dot F/F$, obtaining $\dot F\ddot E - E\dot F\ddot F/F = -\gamma F^{-1}\dot F\,d[t^{-1}(t-2)^{-1}F]/dt$. Also, multiplying (5.4) by $F\ddot F$ we get $\dot E\ddot F - E\dot F\ddot F/F = -\gamma t^{-1}(t-2)^{-1}\ddot F$. Subtracting the last two relations, we see that $(\dot F\ddot E - \dot E\ddot F)/\gamma$ coincides with $L^{-1}\ddot F - F^{-1}\dot F\,d[F/L]/dt$ for $L = t(t-2)$, which, for any given C^2 functions F, L of the real variable t, obviously equals $L^{-2}F\,d[L\dot F/F]/dt$ wherever $FL \neq 0$. Hence $\dot F\ddot E - \dot E\ddot F = \gamma L^{-2}F\,d[L\dot F/F]/dt$ with $L = t(t-2)$. Let us now divide both sides by $(\dot F)^2$ and successively replace: $d[L\dot F/F]/dt$ by $mt(t-2)/(t-1)^2$ (noting that $L\dot F/F = \varLambda(t)/(t-1)$, cf. (5.5.i), and using (5.1.b)), then $\dot F$ by the expression provided by (5.5.i), F by its description in (2.1), L by $t(t-2)$ and, finally, γ by $\binom{2m}{m}$. This yields (11.1) (also at $t \in \{0,2\}$, as both sides are rational functions of t).

Remark 11.2. Let F, E be as in (2.1) for an integer $m \geqslant 2$, and let $(\ )\dot{} = d/dt$. By (11.1) and (5.1.b), $\dot E/\dot F$ then has a nonzero derivative everywhere in $\mathbf{R} \smallsetminus \{0,1\}$. The values/limits of $\dot E/\dot F$ at $\pm\infty, 0, 1$ are listed below. We use the notations of §6, with slanted arrows indicating, again, the monotonicity type of $\dot E/\dot F$.

value or limit at:	$-\infty$		0		1		$+\infty$
for $\dot E/\dot F$ (m even):	1	$\nearrow$	$+\infty\!\downarrow\!-\infty$	$\nearrow$	0	$\nearrow$	1
for $\dot E/\dot F$ (m odd):	1	$\searrow$	$-\infty\!\uparrow\!+\infty$	$\searrow$	0	$\nearrow$	1

(11.2)

In fact, the rational function $\dot E/\dot F$ must have some limits at $\pm\infty$. By l'Hospital's rule, they coincide with those of E/F in §6. (Both E, F have infinite limits at $\pm\infty$, cf. §6.) The limits of $\dot E/\dot F$ at 1 and 0 are easily found using (2.1) and (5.1): $\dot F$ has a pole at 1, and $\dot E$ does not, while $\dot F$ has at 0 a zero of order $2m-2$, greater than the order of a zero at 0 for the degree $m-1$ polynomial $\dot E$.

Proposition 11.3. *Given an integer $m \geqslant 2$, let $Q \in \mathbf{V} \smallsetminus \{0\}$, with $\mathbf{V}$ as in* (2.4)*, satisfy condition* (2.5.b) *on a nontrivial closed interval I with $0 \notin I$ and $1 \notin I$. Then Q must also satisfy conditions* (2.5.a), (2.5.c), (2.5.d).

Proof. Since $1 \notin I$, (2.5.a) follows. Let us now suppose that all the assumptions hold, yet, contrary to our claim, one of conditions (2.5.c), (2.5.d) fails. In view of (2.5.b), the function H with (10.1) then not only vanishes at both endpoints u, v of I, but, in addition, its derivative $\dot H = dH/dt$ is zero at one of the endpoints (if (2.5.d) fails), or $H = 0$ at some interior point of I (if (2.5.c) fails; note that, to evaluate $\dot H$ at an endpoint, e.g., u, we may treat the $(t-1)$ factor in (10.1) like a nonzero constant, since $u \neq 1$ and $Q(u) = 0$). In either case, Rolle's theorem gives $\dot H = 0$ at two distinct points of I. On the other hand, by (10.2) and Lemma 10.1, $\dot H = 0$ at precisely those t at which $\dot E/\dot F = -C/B$. Note that I is contained in one of the intervals $(-\infty,0)$, $(0,1)$, $(1,\infty)$, and hence, according to (6.1), F is strictly monotone on I, so that $B \neq 0$ by Lemma 10.1; however, for a similar reason, $\dot E/\dot F$ is strictly monotone on I (see Remark 11.2), and so it cannot assume the value $-C/B$ twice. This contradiction completes the proof. ∎

12. Conditions (2.5.c), (2.5.d) with an endpoint at 1

The following lemma lists some obvious facts that will help us understand which integers $m \geqslant 2$ and nontrivial closed intervals I containing 1 as an endpoint have the property that a function $Q \in \mathbf{V} \smallsetminus \{0\}$ with (2.5.b) on I also satisfies conditions (2.5.c), (2.5.d). Cf. also Lemma 9.1.

For most of our discussion, the symbol E has stood for the function appearing in (2.1) with a fixed integer $m \geqslant 2$. The following obvious lemma, however, is an exception, as we allow E to be much more general.

Lemma 12.1. *Let $E : \mathbf{R} \to \mathbf{R}$ be any function. For $A, B \in \mathbf{R}$ with $B \neq 0$, and $t \in \mathbf{R}$, let us set*

$$Q(t) \,=\, (t-1)\,[A + BE(t)]\,. \tag{12.1}$$

i) *If $u \in \mathbf{R} \smallsetminus \{1\}$, condition $Q(u) = 0$ holds if and only if $E(u) = -A/B$.*
ii) *Given $t, u \in \mathbf{R}$ with $E(u) = -A/B$, we have $Q(t) = 0$ if and only if $t = 1$ or $E(t) = E(u)$.*
iii) *If E is of class C^1 and $E(1) = 0$, while $u \in \mathbf{R} \smallsetminus \{1\}$ and $E(u) = -A/B$, then*
   a) *$\dot{Q}(1) = 0$ if and only if $E(u) = 0$. Here and in* (b),(c), *$(\ )^{\boldsymbol{\cdot}}$ stands for d/dt.*
   b) *$\dot{Q}(u) = 0$ if and only if $\dot{E}(u) = 0$.*
   c) *$\dot{Q}(1) + \dot{Q}(u) = 0$ if and only if $\dot{\varSigma}(u) = 0$, where $\varSigma(t) = (t-1)^{-1}E(t)$ for $t \neq 1$.*

Remark 12.2. Let $\mathbf{V}$ be the space (2.4) with a fixed integer $m \geqslant 2$. For any nontrivial closed interval I, a function $Q \in \mathbf{V} \smallsetminus \{0\}$ satisfying (2.5.b) on I exists and is unique up to a factor (see Lemma 9.1). In the case where the endpoints of I are $v = 1$ and $u \neq 1$, this Q is given by (12.1) with any constants $B \neq 0$ and A chosen so that $E(u) = -A/B$, for E as in (2.1).

In fact, (12.1) implies (2.3), while, by Lemma 12.1(i), such a choice of A, B gives (2.5.b).

The next section comprises facts we need in order to apply Lemma 12.1 to $E(t)$ and $Q(t)$ given by (2.1) and (2.3). The eventual conclusions about functions $Q \in \mathbf{V}$ satisfying conditions (2.5.b), (2.5.c), (2.5.d) on intervals I with an endpoint at 1 will be presented later; see (iii) in §16.

13. Monotonicity properties of the function E with (2.1)

Let $\dot{E} = dE/dt$ with $E(t)$ as in (2.1) for an integer $m \geqslant 2$. By (5.6.i) – (5.6.ii), $E(0) < 0$ and $\dot{E}(0) = E(1) = 0$. The following claims will be verified in §14:

If m is even, then $\dot{E}(t) \neq 0$ for every $t \in \mathbf{R} \smallsetminus \{0\}$ and there exists a unique $\tilde{z} \in \mathbf{R} \smallsetminus \{1\}$ with $E(\tilde{z}) = 0$. This unique $\tilde{z}$ is negative. With the notations and conventions of §6, we then have

$$\begin{array}{l|ccccccccc} \text{value or limit at:} & -\infty & & \tilde{z} & & 0 & & 1 & & +\infty \\ \hline \text{for } E \ (m \text{ even}): & +\infty & \searrow & 0 & \searrow & E(0) & \nearrow & 0 & \nearrow & +\infty \end{array} \tag{13.1}$$

If m is odd, there exist unique numbers $\tilde{z}, \tilde{w} \in \mathbf{R} \smallsetminus \{0\}$ with $E(\tilde{z}) = E(0)$ and $\dot{E}(\tilde{w}) = 0$. They satisfy the relations $\tilde{z} < \tilde{w} < 0$ and $E(0) < E(\tilde{w}) < 0$. Also, $\dot{E} \neq 0$ everywhere in $\mathbf{R} \smallsetminus \{\tilde{z}, \tilde{w}, 0, 1\}$, and, in the notations of §6,

$$\begin{array}{l|ccccccccccc} \text{value or limit at:} & -\infty & & \tilde{z} & & \tilde{w} & & 0 & & 1 & & +\infty \\ \hline \text{for } E \ (m \text{ odd}): & -\infty & \nearrow & E(0) & \nearrow & E(\tilde{w}) & \searrow & E(0) & \nearrow & 0 & \nearrow & +\infty \end{array} \tag{13.2}$$

14. Proofs of the claims made in §13

According to Remark 11.2, if m is even, $\dot{E}/\dot{F}$ is positive on $(-\infty, 0) \cup (1, \infty)$ and negative on $(0, 1)$, while, if m is odd, $\dot{E}/\dot{F}$ vanishes at a unique $\tilde{w} \in (-\infty, 0)$, is positive on $(-\infty, \tilde{w}) \cup (0, 1) \cup (1, \infty)$,

and is negative on $(\tilde{w},0)$. Combined with the signs of $\dot{F}$ on the individual intervals (cf. the slanted arrows for F in §6), this gives the required signs of $\dot{E}$, that is, slanted arrows for E in §13. (As (2.2) gives $\dot{E}(t) = \varSigma(t) + (t-1)\dot{\varSigma}(t)$, we have $\dot{E}(1) = \varSigma(1) > 0$ by Remark 5.1.)

Since E is a nonconstant polynomial, its limits at $\pm\infty$ are infinite, with the signs required in §13 (see the slanted arrows). This proves all statements except for those involving $\tilde{z}$ and the relation $E(0) < E(\tilde{w}) < 0$ for odd m. However, $E(0) < E(\tilde{w})$ as E is decreasing on $[\tilde{w},0]$, so that the already-established monotonicity pattern of E gives, for all m, the existence and uniqueness of $\tilde{z}$ along with $\tilde{z} < 0$ (m even) or $\tilde{z} < \tilde{w}$ (m odd). Finally, if m is odd, $E(\tilde{w}) < 0$ by (7.1.d).

15. Conditions equivalent to (2.5.c) − (2.5.d)

Lemma 15.1. *Given an integer $m \geqslant 2$, let the functions E, F, H be given by* (2.1) *and* (10.2) *for any fixed $u, v \in \mathbf{R}$ with $u \neq 1 \neq v \neq u$. If the restrictions of E and H to the interval $(-\infty, 1)$ of the variable t are treated as continuous functions of the new variable $F \in \mathbf{R}$, differentiable on $\mathbf{R} \smallsetminus \{0\}$, cf.* §6, *then*

$$\text{i)}\quad H = A + BE + CF\,, \qquad\qquad \text{ii)}\quad dH/dF = C + B\,dE/dF \tag{15.1}$$

wherever $F \neq 0$, with A, B, C defined by (10.3). *In addition, $d^2E/dF^2 < 0$ wherever $F \neq 0$, which, if $B < 0$, amounts to $d^2H/dF^2 > 0$.*

In fact, (10.2.i) and (10.3) give (15.1.i), and hence (15.1.ii), while $d^2E/dF^2 < 0$ by (5.1) and (11.1), so that (15.1.ii) with $B < 0$ yields $d^2H/dF^2 > 0$.

Proposition 15.2. *Given an integer $m \geqslant 2$ and a nontrivial closed interval $I \subset \mathbf{R}$ with the endpoints u, v, such that 0 is an interior point of I and $1 \notin I$, let $F, E, \mathbf{V}$ be as in* (2.1), *(2.4), and let $Q \in \mathbf{V} \smallsetminus \{0\}$ satisfy on I condition* (2.5.b).

With A, B, C depending on u, v as in (10.3) *and with $\dot{F} = dF/dt$, $\dot{E} = dE/dt$, we then have $B\dot{F}(u)\dot{F}(v) \neq 0$. In addition, Q and I satisfy* (2.5.a), *(2.5.c) and* (2.5.d) *if and only if one of the following two conditions holds*:

i) *$\dot{E}(u)/\dot{F}(u) < -C/B < \dot{E}(v)/\dot{F}(v)$, with the endpoints u, v of I switched, if necessary, to ensure that $(-1)^m(u-v) > 0$, or*

ii) *$A/B > -E(0)$.*

Proof. Our assumption on I states that $u < 0 < v < 1$ or $v < 0 < u < 1$. As in (10.2.iii), let us set $F_0 = F(u)$, $F_1 = F(v)$. As $(-1)^m\dot{F} < 0$ on $(-\infty,0) \cup (0,1)$ and $F(0) = 0$, cf. (6.1), the additional assumption $(-1)^m(u-v) > 0$ made in (i) now gives $F_0 < 0 < F_1$, while, by (10.3), $B < 0$ and $B\dot{F}(u)\dot{F}(v) \neq 0$.

Let H be the function defined in (10.1) (that is, (10.2)) for our u, v. Using the coordinate change $t \mapsto F = F(t)$, as in Lemma 15.1, we may treat E and H not as functions of $t \in (-\infty, 1)$, but rather as functions of the variable $F \in \mathbf{R}$, continuous at $F = 0$ and of class C^∞ everywhere else. Then, as $\dot{E}/\dot{F} = dE/dF$,

$$\text{Condition (i) means that } dH/dF \text{ is positive at } F = F_0 \text{ and negative at } F = F_1, \tag{15.2}$$

by (15.1.ii) with $B < 0$, while, by (15.1.i) with $F(0) = 0$ (cf. (2.1)) and $B < 0$,

$$\text{Inequality (ii) states that } H < 0 \text{ at } F = 0\,. \tag{15.3}$$

Also, since H and Q are, up to a nonzero factor, related by (10.1),

$$\begin{array}{l}\text{Conditions (2.5.c), (2.5.d) for our pair } Q, I \text{ mean that } H \neq 0 \text{ at all } F\\ \text{with } F_0 < F < F_1\,, \text{ while } dH/dF \neq 0 \text{ at } F = F_0 \text{ and } F = F_1\,.\end{array} \tag{15.4}$$

In fact, to see if $dH/dF = (dH/dt)/(dF/dt)$ is zero or not, we only need to apply d/dt to the $Q(t)$ factor in (10.1), as $Q(u) = Q(v) = 0$, while, by (5.1), $dF/dt \neq 0$ when $0 \neq t \neq 1$.

Let Q and I now satisfy (2.5.a), (2.5.c) and (2.5.d). By (15.4), H must be nonzero throughout the whole interval (F_0, F_1) of the variable F, and $dH/dF \neq 0$ at the endpoints F_0, F_1. Thus, H is positive (or, negative) on (F_0, F_1), which yields the clause about H in (15.2) (or, (15.3), and hence (i) or, respectively, (ii).

Conversely, let us assume (i) or (ii). In case (i), using (15.2) and the inequality $d^2H/dF^2 > 0$ whenever $F \neq 0$ (Lemma 15.1), we see that dH/dF is positive for all F with $F_0 < F < 0$, and negative if $0 < F < F_1$. As $H = 0$ at both $F = F_0$ and $F = F_1$, this in turn gives $H > 0$ at every F with $F_0 < F < F_1$. Since, by (15.2), $dH/dF \neq 0$ at the endpoints F_0, F_1, conditions (2.5.c) and (2.5.d) for Q and I follow in view of (15.4) (while (2.5.a) is obvious as $1 \notin I$).

Finally, let us consider the remaining case (ii). By (15.3), we then have $H < 0$ at $F = 0$. It now follows that $dH/dF < 0$ at $F = F_0$ and $H \neq 0$ (so that $H < 0$) everywhere in the interval $(F_0, 0)$. In fact, if either of these claims failed, we could find F_2, F_3 with $F_0 \leqslant F_2 \leqslant F_3 < 0$, $dH/dF \geqslant 0$ at $F = F_2$, and $H = 0$ at $F = F_3$. (Specifically, we set $F_2 = F_3 = F_0$ if $dH/dF \geqslant 0$ at $F = F_0$; while, if $H = 0$ at some F_3 in $(F_0, 0)$, we can use Rolle's theorem to select F_2.) Since dH/dF is a strictly increasing function of $F \in (F_0, 0)$ (as $d^2H/dF^2 > 0$ by Lemma 15.1), we have $dH/dF > 0$ on $(F_3, 0)$, which is not possible as $H = 0$ at $F = F_3$ and $H < 0$ at $F = 0$. A completely analogous argument shows that, if (ii) holds, we must have $dH/dF > 0$ at $F = F_1$ and $H \neq 0$ (that is, $H < 0$) everywhere in the interval $(0, F_1)$. In other words, by (15.4), assuming (ii) we obtain (2.5.a), (2.5.c) and (2.5.d) as well. This completes the proof. $\square$

16. Positivity

Let $\mathbf{V}$ be the space (2.4) for a given integer $m \geqslant 2$. According to Lemma 9.1, for every nontrivial closed interval I, a function $Q \in \mathbf{V} \smallsetminus \{0\}$ satisfying on I condition (2.5.b) exists and is unique up to a constant factor. We will say that I satisfies the *positivity condition* if, in addition to (2.5.b), we also have (2.5.a), (2.5.c) and (2.5.d) for some, or any, such Q.

Now let $m \geqslant 2$ be fixed. The discussion in the preceding sections has determined that for any given nontrivial closed interval I the positivity condition holds

i) Always, if I contains neither 0 nor 1 (Proposition 11.3).

ii) Never, if I contains 1 as an interior point or 0 as an endpoint. This is clear from Remark 2.1 or, respectively, the fact that, by (9.2.a), condition (2.5.b) for the endpoint 0 contradicts condition (2.5.d).

iii) When I contains 1 as an endpoint: if and only if the other endpoint lies in $(-\infty, \tilde{z}) \cup (0, \infty)$ for the number $\tilde{z} < 0$ defined in §13. (See below.)

iv) When I contains 0 as an interior point and does not contain 1 at all: if and only if the endpoints u, v of I satisfy (i) or (ii) in Proposition 15.2.

Only (iii) still requires an explanation. Namely, as $1 \in I$, condition (2.5.a) gives $C = 0$, so that (2.3) becomes (12.1), and the remaining endpoint $u \neq 1$ determines Q up to a factor via Lemma 12.1(i). Hence, by Lemma 12.1(ii), (iii)a),b), conditions (2.5.d) and (2.5.c), in addition to (2.5.b) and (2.5.a), amount to requiring that $E(u)\dot{E}(u) \neq 0$ and $E(t) \neq E(u)$ for all t in the open interval connecting 1 and u. The claims made in (iii) now are trivial consequences of the descriptions in §13 of the monotonicity intervals for E and the roots of E and $\dot{E}$.

17. Factorization of (2.5.e)

Lemma 17.1. *Let Φ, φ be polynomials in two variables u, v. Then Φ is divisible by φ if*

i) $\deg\varphi = 1$, *while* $\Phi = 0$ *wherever* $\varphi = 0$, *or*

ii) $\varphi = (u-v)^3$ *and* Φ *is antisymmetric, while* $\partial\Phi/\partial v = 0$ *wherever* $u = v$.

In fact, (i) is clear if we use new affine coordinates ξ, η with $\varphi = \xi$. Next, for Φ as in (ii), $\tilde\Phi = \Phi/(v-u)$ is, by (i), a symmetric polynomial with $\tilde\Phi = \partial\Phi/\partial v$ wherever $u = v$. Thus, for the new coordinates ξ, η given by

$$u = \xi + \eta\,, \quad v = \xi - \eta\,, \qquad \text{that is,} \quad \xi = (u+v)/2\,, \quad \eta = (u-v)/2\,, \tag{17.1}$$

$\tilde\Phi$ is even in η (and so it is a polynomial in ξ and η^2) and vanishes wherever $\eta = 0$ (due to the assumption on $\partial\Phi/\partial v$). Hence, by (i), $\tilde\Phi$ is divisible by η^2.

Lemma 17.2. *With F, E as in* (2.1) *for an integer $m \geqslant 2$, and $(\,)^\cdot = d/dt$, let*

$$\Pi(u,v) \,=\, (v-1)\Big\{[F(v)-F(u)]\dot E(v) \,-\, [E(v)-E(u)]\dot F(v)\Big\}. \tag{17.2}$$

Then, for A, B depending on u, v as in (10.3), $u(u-2)v(v-2)[\Pi(u,v)-\Pi(v,u)]$ *equals*

$$2(uv-u-v) \ \text{ times } \ [muv-(2m-1)(u+v-2)]A \,+\, (2m-1)(u+v-2)B\Sigma(0)\,. \tag{17.3}$$

In fact, by (17.2) and (5.5), $u(u-2)v(v-2)[\Pi(u,v)-\Pi(v,u)] = [v(v-2)\Lambda(u)+u(u-2)\Lambda(v)]A + 2(2m-1)[u(u-2)(v-1)+v(v-2)(u-1)]B\Sigma(0)$, with Λ as in (5.1). This proves our claim, since $v(v-2)\Lambda(u)+u(u-2)\Lambda(v) = 2m(uv-u-v)[uv-(2-1/m)(u+v-2)]$ and

$$u(u-2)(v-1) \,+\, v(v-2)(u-1) \,=\, (uv-u-v)(u+v-2)\,. \tag{17.4}$$

Lemma 17.3. *For any integer $m \geqslant 2$, there exists a unique symmetric polynomial T in the variables u, v such that, for Π as in* (17.2),

$$(u-1)^m(v-1)^m\,[\Pi(u,v)-\Pi(v,u)] \,=\, 2(v-u)^3(uv-u-v)\,T(u,v)\,. \tag{17.5}$$

Proof. Multiplication by $(u-1)^m(v-1)^m$ turns (17.3), as well as $\Pi(u,v)$ and $\Pi(v,u)$, into polynomials in u, v (cf. (17.2) and (2.1), (5.1)). Lemmas 17.1 – 17.2 now show that the left-hand side of (17.5) is a polynomial divisible by $uv-u-v$.

On the other hand, Π in (17.2) clearly vanishes whenever $u = v$. Also, the $\partial\Phi/\partial v$ clause in Lemma 17.1(ii) is satisfied both by $\Phi = \Pi$ and $\Phi(u,v) = \Pi(v,u)$. (One verifies this without evaluating $\ddot E, \ddot F$, since, in the Leibniz-rule expression for the partial derivative, only the factors $F(v)-F(u)$ and $E(v)-E(u)$ need to be differentiated, as they vanish when $u = v$.) In view of Lemma 17.1(ii) and the last paragraph, the polynomial on the left-hand side of (17.5) divided by $uv-u-v$ is still divisible by $(v-u)^3$. This completes the proof. $\square$

Lemma 17.4. *Given an integer $m \geqslant 2$ and a nontrivial closed interval $I \subset \mathbf{R}$ with the endpoints $v = 1$ and $u \neq 1$, let $\dot\Sigma = d\Sigma/dt$ for Σ defined by* (2.2)*. Also, let $\mathbf{V}$ be the space* (2.4) *and let $Q \in \mathbf{V} \smallsetminus \{0\}$ be a function, unique up to a factor, which satisfies* (2.5.b) *on I, cf.* Lemma 9.1. *Condition $\dot\Sigma(u) = 0$ then is necessary and sufficient for Q and I to satisfy* (2.5.e).

In fact, as 1 is an endpoint of I, (2.5.b) gives (2.5.a) and we have (2.3) with $C = 0$, that is, (12.1) for some $B \neq 0$ and A with $E(u) = -A/B$ (see Lemma 12.1(i)). Conditions $\dot\Sigma(u) = 0$ and (2.5.e) now are equivalent by Lemma 12.1(iii)c).

Theorem 17.5. *Given an integer $m \geqslant 2$ and a nontrivial closed interval I with endpoints u, v, let a function $Q \in \mathbf{V} \smallsetminus \{0\}$, with $\mathbf{V}$ as in* (2.4)*, satisfy on I condition* (2.5.b)*, that is, $Q(u) = Q(v) = 0$. By* Lemma 9.1*, such Q exists and is unique up to a constant factor. Also, let $(\,)^\cdot = d/dt$.*

Then Q *satisfies* (2.5.e), *that is,* $\dot{Q}(u)+\dot{Q}(v)=0$, *if and only if, for the polynomials* $\varSigma, T$ *defined in* (2.2) *and* Lemma 17.3, *one of the following three cases occurs*:

$$\begin{array}{lll} \text{a)} & u=1 \ \text{ and } \ \dot{\varSigma}(v)=0\,, \qquad \text{or} \qquad v=1 \ \text{ and } \ \dot{\varSigma}(u)=0\,. & \\ \text{b)} & uv\,=\,u\,+\,v\,. & \\ \text{c)} & T(u,v)\,=\,0 \qquad\quad \text{and} \qquad u\neq 1\neq v\,. & \end{array} \tag{17.6}$$

Proof. If one of u,v equals 1, our assertion, stating in this case that condition (2.5.e) is equivalent to (17.6.a), is nothing else than Lemma 17.4. Let us therefore assume that $u\neq 1\neq v$, and consider a function $Q\in\mathbf{V}\smallsetminus\{0\}$ satisfying on I condition (2.5.b). Using the subscript convention (10.2.iii) (also for functions of t other than E, F), we can rewrite (2.5.e) as $0=\dot{Q}_0+\dot{Q}_1=(u-1)\dot{H}_0+(v-1)\dot{H}_1$ (see (10.1); by (2.5.b), $H_0=H_1=0$). Thus, (2.5.e) states that

$$(u-1)[(F_1-F_0)\dot{E}_0-(E_1-E_0)\dot{F}_0]+(v-1)[(F_1-F_0)\dot{E}_1-(E_1-E_0)\dot{F}_1]=0, \tag{17.7}$$

cf. (10.2.ii), and so $\varPi(v,u)=\varPi(u,v)$, for $\varPi$ given by (17.2). As $u\neq 1\neq v\neq u$, (17.5) shows that, when $u\neq 1\neq v$, (2.5.e) holds if and only if we have (17.6.b) or (17.6.c). Since, conversely, (17.6.b) implies that $u\neq 1\neq v$, this completes the proof. □

Conditions (17.6.b), (17.6.c) are not mutually exclusive: when m is odd, they may occur simultaneously, with $u\neq v$. See Lemma 25.2(b) in §25.

18. First subcase of (2.5.e): condition (17.6.a)

Let $m\geqslant 2$ be a fixed integer. We will now find all pairs Q,I formed by a function $Q\in\mathbf{V}\smallsetminus\{0\}$, with $\mathbf{V}$ as in (2.4), and a nontrivial closed interval I such that $1\in I$ and Q satisfies on I the boundary conditions (2.5.b), (2.5.e). As we show in Proposition 18.3 below, such Q,I exist if and only if m is odd, and then they are unique up to multiplication of Q by a nonzero constant.

With $(\ )^\cdot=d/dt$, (2.2) gives $(t-1)\dot{\varSigma}(t)=\dot{E}(t)-\varSigma(t)$ for all real t. Multiplying this by $t(t-1)(t-2)$ and using (5.5.ii), (2.2), (5.1.b), we easily obtain

$$t(t-1)(t-2)\dot{\varSigma}(t)\;=\;[(m-1)(t-2)^2+2]\,\varSigma(t)\;-\;2(2m-1)\,\varSigma(0). \tag{18.1}$$

Remark 18.1. We will repeatedly use the following obvious fact. If the derivative of a C^1 function $\varPhi$ on an interval $\mathcal{I}$ is positive wherever $\varPhi=0$ in $\mathcal{I}$, then there exists at most one $t\in\mathcal{I}$ with $\varPhi(t)=0$. If such t exists, then $\varPhi<0$ on $(-\infty,t)\cap\mathcal{I}$ and $\varPhi>0$ on $(t,\infty)\cap\mathcal{I}$.

Proposition 18.2. *Given an integer* $m\geqslant 2$, *let* $\varSigma$ *be the polynomial in* (2.2), *and let* $X(t)=2(2m-1)\,\varSigma(0)\,[(m-1)(t-2)^2+2]^{-1}$.

a) *If* m *is even, then* $\dot{\varSigma}=d\varSigma/dt$ *has no real roots.*

b) *If* m *is odd,* $\dot{\varSigma}$ *has exactly one real root* s, *and that root is negative.*

c) $X(t)>0$ *for all* $t\in\mathbf{R}$. *For any* $t<0$, *we have* $\dot{\varSigma}(t)=0$ *if and only if* $\varSigma(t)=X(t)$.

Proof. The function X is positive since $\varSigma(0)>0$ (cf. (5.6.i)), and so (18.1) yields (c). Next, one and only one of the following two conditions must hold:

i) $\dot{\varSigma}>0$ and $\varSigma<X$ everywhere in $(-\infty,0)$.

ii) There exists a unique $s\in(-\infty,0)$ with $\dot{\varSigma}(s)=0$, and then $\varSigma>X$ on $(-\infty,s)$, while $\varSigma(s)=X(s)$ and $\varSigma<X$ on $(s,0)$.

In fact, the definition of X easily gives

a) $\dot{X}=dX/dt$ is positive everywhere in $(-\infty,0]$. b) $X(0)=\varSigma(0)$ and $\dot{X}(0)<\dot{\varSigma}(0)$,

the last inequality being clear as $\dot{X}(0) = (2m-2)\varSigma(0)/(2m-1)$, while $\dot{\varSigma}(0) = \varSigma(0) > 0$ (see (5.6.i)). Now let $\varPhi = X - \varSigma$. By (d) and (c), $\dot{\varPhi} > 0$ wherever $\varPhi = 0$ in $(-\infty, 0)$. Thus, according to Remark 18.1, if some $s \in (-\infty, 0)$ has $\dot{\varSigma}(s) = 0$ (that is, $\varPhi(s) = 0$, cf. (c)), then such s is unique, while $\varPhi < 0$ on $(-\infty, s)$ and $\varPhi > 0$ on $(s, 0)$. This yields (ii). However, if no such s exists, we have $\varPhi\dot{\varSigma} \neq 0$ everywhere in $(-\infty, 0)$. Hence $\dot{\varSigma} > 0$ on $(-\infty, 0]$, as $\dot{\varSigma}(0) > \dot{X}(0) > 0$ by (d) and (e). Also, by (e), $\dot{\varPhi}(0) < 0 = \varPhi(0)$, so that $\varPhi(t) > 0$ for all $t < 0$ close to 0. Thus, $\varPhi > 0$ on $(-\infty, 0)$, and so $\varSigma < X$ on $(-\infty, 0)$, which gives (i).

In case (i), Remark 5.1 shows that $\dot{\varSigma}$ has no real roots, and so m is even (as $\deg \dot{\varSigma} = m-2$). In the remaining case (ii), $\dot{\varSigma}$ has exactly one real root s (cf. Remark 5.1), and so m must be odd, which is clear since $\varSigma$ is a polynomial of degree $m-1$ with a positive leading coefficient, cf. (2.2), while $\varSigma > X > 0$ on $(-\infty, s)$ by (ii) and (c). This yields (a) and (b), completing the proof. $\square$

Proposition 18.3. *For a fixed integer $m \geqslant 2$, let $F, E, \varSigma, \mathbf{V}$ be given by* (2.1), (2.2), (2.4). *The following two conditions are equivalent*:

i) *There exist a nontrivial closed interval I containing 1 and a function $Q \in \mathbf{V} \smallsetminus \{0\}$ satisfying on I conditions* (2.5.b), (2.5.e).
ii) *m is odd.*

If m is odd, Q, I in (i) *are unique up to multiplication of Q by a nonzero constant. Specifically, the endpoints of I then are 1 and the unique $s \in (-\infty, 0)$ with $\dot{\varSigma}(s) = 0$, cf.* Proposition 18.2(b), *while Q is given by* (12.1) *for any constants $B \neq 0$ and A with $E(s) = -A/B$.*

In fact, if m is even, $\dot{\varSigma}(u) \neq 0$ for all $u \in \mathbf{R}$ (Proposition 18.2(a)); thus, by Remark 2.1 and Lemma 17.4, no pair Q, I has the properties listed in (i).

Conversely, if m is odd, Lemma 17.4, Proposition 18.2(b) and Remark 2.1 imply both the existence of a pair Q, I with (i), and uniqueness of the interval I, the endpoints of which must be those described in our assertion. Finally, uniqueness of Q up to a factor and its required form are immediate from Remark 12.2.

19. Failure of positivity in case (17.6.a)

In this section we show (Proposition 19.2) that case (17.6.a) of condition (2.5.e) cannot occur simultaneously with the positivity condition introduced in §16.

Lemma 19.1. *Given an odd integer $m \geqslant 3$, let s be the unique real number with $\dot{\varSigma}(s) = 0$ for $\varSigma$ as in* (2.2), *cf.* Proposition 18.2(b). *Then, with $\tilde{z}, \tilde{w}$ as in* §13,

$$\begin{array}{rlll} \text{i)} & s \leqslant -\eta\,, & \text{where } \eta = (2m-3)/[3(m-2)]\,, & \\ \text{ii)} & \varSigma(t) \geqslant \varSigma(s) > 0 & \text{for all } t \in \mathbf{R}\,, & \\ \text{iii)} & \tilde{z} \leqslant s < \tilde{w} < 0\,, & \text{and } \tilde{z} < s \text{ if } m > 3. & \end{array} \tag{19.1}$$

Proof. Differentiating the second equality in (2.2) we obtain $\dot{\varSigma}(t) = \sum_{k=2}^{m} a_k t^{k-2}$ with coefficients $a_k > 0$ such that $a_k/a_{k+1} = [1 - 2/(k+1)][1 + (m-1)/(m-k)]$, where both factors in square brackets clearly are increasing positive functions of $k = 2, \ldots, m-1$. Thus, $a_k/a_{k+1} \geqslant a_2/a_3 = \eta$ (cf. (19.1.i)), that is, $a_k \geqslant a_{k+1}\eta$, and so $\dot{\varSigma}(-\eta) = (a_2 - a_3\eta) + (a_4 - a_5\eta)\eta^2 + \ldots + (a_{m-1} - a_m\eta)\eta^{m-3} \geqslant 0$. As $\dot{\varSigma}$ is an odd-degree polynomial with a positive leading coefficient and with a unique root at s, it must be negative on $(-\infty, s)$ and positive on (s, ∞), so that the last inequality proves (19.1.i), and $\varSigma(t) \geqslant \varSigma(s)$ for all $t \in \mathbf{R}$. Furthermore, Proposition 18.2(c), with $t = s$, gives $\varSigma(s) = X(s) > 0$ for $X(t)$ as in Proposition 18.2, which yields the remaining inequality in (19.1.ii).

In the remainder of this proof, all inequalities are strict if $m > 3$. First, $0 \leqslant (2m-5)(m-3) = (2m-3)(m-1) - 6(m-2)$ (since $m \geqslant 3$), so that $\eta \geqslant 2/(m-1)$ for η as in (19.1.i). Now

(19.1.i) yields $s \leqslant -2/(m-1)$. This gives $[(m-1)s+2]s \geqslant 0$ and $[(m-1)s+2]s\varSigma(0) \geqslant 0$ (since $\varSigma(0) > 0$ by (5.6.i)). As $\dot\varSigma(s) = 0$, the right-hand side of (18.1) with $t = s$ vanishes, which, multiplied by $s-1$, reads $[(m-1)(s-2)^2+2]E(s) - 2(2m-1)(s-1)\varSigma(0) = 0$ (see (2.2)). Adding this side-by-side to the last inequality we get $[(m-1)(s-2)^2+2][E(s)+\varSigma(0)] \geqslant 0$. Hence $E(s) \geqslant -\varSigma(0) = E(0)$ (cf. (2.2)). Also, as $\dot E(t) = \varSigma(t) + (t-1)\dot\varSigma(t)$ (by (2.2)) and $\dot\varSigma(s) = 0$, (19.1.ii) gives $\dot E(s) = \varSigma(s) > 0$. Since $\tilde w < 0$ (see §13) and $s < 0$, while $E(s) \geqslant E(0)$ and $\dot E(s) > 0$ (see above), (13.2) yields (19.1.iii). This completes the proof. □

Proposition 19.2. *Given an integer $m \geqslant 2$, there exists no function $Q \in \mathbf{V}$, for $\mathbf{V}$ as in* (2.4), *satisfying* (2.5) *on any nontrivial closed interval I that contains* 1.

In fact, let (2.5.a,b,e) hold for Q and I. By Proposition 18.3, m is odd, and the endpoint of I other than 1 is the unique $s < 0$ with $\dot\varSigma(s) = 0$. However, by (19.1.iii), s fails the "positivity test" in (iii) of §16, that is, Q and I cannot satisfy all four conditions (2.5.a) – (2.5.d).

Remark 19.3. For the moduli curve $\mathcal{C}$ defined at the end of §2 and any point $(u,v) \in \mathcal{C} \smallsetminus \{(0,0)\}$, Proposition 19.2 clearly implies that $1 < u < v$ or $u < v < 1$.

Example 19.4. For $m = 3$, we have $\tilde z = s = -1$ and $\tilde w = -2/3$ in (19.1.iii). In fact, $-1, -1, -2/3$ are the unique negative roots of $E - E(0)$, $\dot\varSigma$ and $\dot E$, since, by (2.2), $\varSigma(t) = t^2+2t+2$ for $m = 3$, and so $E(t) = t^3+t^2-2$, $E(t) - E(0) = t^2(t+1)$, $\dot\varSigma(t) = 2(t+1)$ and $\dot E(t) = t(3t+2)$. Cf. Example 27.4 and the end of §21.

20. Second subcase of (2.5.e): condition (17.6.b)

In this section we explicitly classify all pairs Q, I that satisfy the boundary conditions (2.5.b), (2.5.e) and are of type (17.6.b).

Given an integer $m \geqslant 2$, let F, E be as in (2.1). Then, for all $t \in \mathbf{R} \smallsetminus \{1\}$,

$$\text{i)}\quad F(t^*) = -F(t), \qquad \text{ii)}\quad E(t^*) = E(t) - F(t), \tag{20.1}$$

where t^* is, for any $t \neq 1$, given by either of the two equivalent relations

$$\text{a)}\quad t^* = t/(t-1)\,, \qquad \text{b)}\quad t^*-1 = 1/(t-1)\,. \tag{20.2}$$

In fact, (20.1.i) is immediate from (2.1), and (20.1.ii) is easily verified by induction on m, using (5.2). Since (20.2.a) is equivalent to (20.2.b), we obtain

Lemma 20.1. *The assignment $u \mapsto u^*$ with $u^* = u/(u-1)$ is an involution of $\mathbf{R} \smallsetminus \{1\}$, decreasing both on $(-\infty, 1)$ and on $(1, \infty)$. It interchanges the interval $(1, 2]$ with $[2, \infty)$, and $(-\infty, 0]$ with $[0, 1)$, while its fixed points are 0 and 2.*

Lemma 20.2. *Given an integer $m \geqslant 2$ and a real number $u \notin \{0, 1, 2\}$, let $v = u^*$, where $u^* = u/(u-1)$. Formula* (10.1) *then defines a function $Q \in \mathbf{V} \smallsetminus \{0\}$, for $\mathbf{V}$ as in* (2.4), *for which* (2.5.a), (2.5.b) *and* (2.5.e) *hold on the nontrivial closed interval I with the endpoints u, v. Moreover, u, v satisfy condition* (17.6.b).

In fact, $u \neq v$ since $u \notin \{0, 1, 2\}$ (cf. Lemma 20.1). Thus, in view of (10.1), $Q \in \mathbf{V} \smallsetminus \{0\}$ and (2.5.b) holds. Also, by Lemma 20.1, $1 \notin I$, which implies (2.5.a). Finally, relation $v = u^*$ is nothing else than (17.6.b). Theorem 17.5 now gives (2.5.e) for Q and I.

Theorem 20.3. *Let $m \geqslant 2$ be a fixed integer. Assigning to each real number u with $u < 0$ or $1 < u < 2$ the interval $I = [u, v]$ with $v = u/(u-1)$ and the function Q defined up to a factor by* (10.1), *with E, F as in* (2.1), *we obtain a bijective correspondence between*

i) *The subset* $(-\infty,0)\cup(1,2)$ *of* $\mathbf{R}$, *and*

ii) *The set of all equivalence classes, modulo multiplication of* Q *by nonzero scalars, of pairs* Q,I *formed by a nontrivial closed interval* $I\subset\mathbf{R}$ *and a function* $Q\in\mathbf{V}\smallsetminus\{0\}$, *with* $\mathbf{V}$ *as in* (2.4), *such that* Q *and* I *satisfy conditions* (2.5.b) *and* (17.6.b), *i.e.,* $Q(u)=Q(v)=0$ *and* $uv=u+v$, *where* u,v *are the endpoints of* I.

Proof. That I and Q described above have the properties listed in (ii) is clear from Lemma 20.2. Injectivity of our assignment is obvious since Q,I obtained from the given u determine u uniquely (as the lower endpoint of I).

To show that the assignment is surjective, let us fix Q,I as in (ii), and let $I=[u,v]$. Since u,v satisfy (17.6.b), we have $(u-1)v=u$, so that $u\neq 1$ and $v=u^*$ (notation of (20.2)). As $u<v$, this gives $u<0$ or $1<u<2$ (cf. Lemma 20.1). Lemma 10.1 now shows that Q is, up to a factor, given by (2.3) with (10.3), that is, by (10.1). Thus, the equivalence class of Q,I is the image of u under our assignment, which completes the proof. $\square$

21. Another rational function

Given an integer $m\geqslant 2$, we define a rational function G of the variable t by

$$G\,=\,E\,-\,F/2\,,\qquad\text{with}\quad E,F\quad\text{as in (2.1).}\tag{21.1}$$

For any $t\in\mathbf{R}\smallsetminus\{1\}$ we then have, with $t^*=t/(t-1)$ as in (20.2) and $(\ )^{\cdot}=d/dt$,

$$\begin{array}{ll}\text{a)} & G(t^*)\,=\,G(t)\,,\qquad \text{b)}\quad \dot G(t^*)/\dot F(t^*)\,=\,-\dot G(t)/\dot F(t)\quad\text{if }\ t\neq 0,\\ \text{c)} & t(t-1)(t-2)\dot G(t)\,=\,\varLambda(t)G(t)\,-\,2(2m-1)(t-1)\varSigma(0),\\ \text{d)} & G(0)\,=\,-\varSigma(0)\,<\,0,\quad \dot G(0)\,=\,0,\quad \text{e)}\quad (2m-3)\ddot G(0)\,=\,m\varSigma(0)\,>\,0,\\ \text{f)} & G(2)\,=\,(2m-1)\varSigma(0),\qquad \dot G(2)\,=\,0,\end{array}\tag{21.2}$$

where $\varLambda$ is given by (5.1.b). In fact, (21.2.a), which is immediate from (21.1) and (20.1), states that G restricted to either of the intervals $(-\infty,1)$, $(1,\infty)$ is an even function of the new variable $F\in(-\infty,\infty)$, cf. §6 and (20.1.i). As the derivative $dG/dF=\dot G/\dot F$ of the even function G is odd, (21.2.b) follows. (Also, $\dot F\neq 0$ on $\mathbf{R}\smallsetminus\{0,1\}$ by (5.1).) Next, (5.5) and (21.1) yield (21.2.c), while (2.1) gives $F(0)=\dot F(0)=0$, and so (21.2.d) is immediate from (5.6). Finally, (21.2.e) and (21.2.f) are easily obtained from (21.2.c) by evaluating it at $t=2$, or differentiating it once/twice at $t=2$ and, respectively, $t=0$.

Note that (20.1) and (21.1) lead to the following special case of (10.3):

$$(A,B,C)=(-2F(u)G(u),\,2F(u),\,-F(u))\quad\text{whenever}\quad u\neq 1\text{ and }v=u^*.\tag{21.3}$$

If m is even, $\dot G=dG/dt$ is negative on $(-\infty,0)$ and there exists a unique negative real number z with $G(z)=0$. The following diagram uses slanted arrows to describes the monotonicity type of G (that is, the sign of $\dot G$) on some specific intervals, and lists the limits of G at their endpoints.

$$\begin{array}{lccccc}\text{value or limit at:} & -\infty & & z & & 0\\ \hline \text{for } G\ (m\text{ even}): & +\infty & \searrow & 0 & \searrow & E(0)\end{array}\tag{21.4}$$

For odd m, there exists unique numbers $z,w\in(-\infty,0)$ with $G(z)=E(0)$ and $\dot G(w)=0$. Moreover, $z<w<0$ and $E(0)=G(z)<G(w)<0$, while $\dot G>0$ on $(-\infty,w)$ and $\dot G<0$ on $(w,0)$. With the same notations as in (21.4),

$$\begin{array}{lccccccc}\text{value or limit at:} & -\infty & & z & & w & & 0\\ \hline \text{for } G\ (m\text{ odd}): & -\infty & \nearrow & E(0) & \nearrow & G(w) & \searrow & E(0)\end{array}\tag{21.5}$$

Remark 21.1. Note the analogy between z, w for G and $\tilde{z}, \tilde{w}$ for E in §13. Also, if m is odd, $z < \tilde{z}$ by (21.5), since $F < 0$ on $(-\infty, 0)$ (see (6.1)), and so at $\tilde{z}$ we have $G = E - F/2 > E = E(0)$.

The above claims, justified in the next section, lead to a general conclusion about the rational function G given by (21.1) with a fixed integer $m \geqslant 2$, which, by (2.1), has just one pole, at 1. Namely, $G \neq 0$ everywhere in $\mathbf{R} \smallsetminus \{1\}$ for odd m, while, if m is even, G has two zeros in $\mathbf{R} \smallsetminus \{1\}$, located at z and $z^* = z/(z-1)$. In fact, as $G(w) < 0$, (21.4) – (21.5) show that $G \neq 0$ everywhere in $(-\infty, 0]$ except at z for even m, while (7.1.c) and (2.1) give $E > F \geqslant 0$ on $[2, \infty)$, so that $E > F/2$ and $G > 0$ on $[2, \infty)$. A similar assertion about the remaining intervals now follows since, by (21.2.a), G is invariant under the involution in Lemma 20.1, which sends $[0, 1)$ and $(1, 2]$ onto $(-\infty, 0]$ and, respectively, $[2, \infty)$.

For $m = 3$ we have $2z = 1 - \sqrt{3} - (2\sqrt{3})^{1/2}$, $6w = 2 - \sqrt{10} - (8\sqrt{10} - 10)^{1/2}$. Thus, $z \approx -1.2966$ and $w \approx -0.8456$. In fact, $2(t-1)^3\,[G(t) - E(0)] = t^2[t^2 - (1+\sqrt{3})(t-1)][t^2 - (1-\sqrt{3})(t-1)]$ by (21.1) and (2.1), and, similarly, since $E(t) = t^3 + t^2 - 2$ (cf. Example 19.4), we obtain $6(t-1)^4\,\dot{G}(t) = t(t-2)[3t^2 - (2+\sqrt{10})(t-1)][3t^2 - (2-\sqrt{10})(t-1)]$.

22. Proofs of (21.4) – (21.5)

According to (6.1), $(-1)^m F(t) \to \infty$ and $2G(t)/F(t) \to 1$ as $t \to -\infty$. (Note that $2G/F = -1 + 2E/F$, cf. (21.1).) Therefore, by (5.1) and (21.1) with $(\ )^\cdot = d/dt$,

$$\begin{array}{rl} \text{i)} & (-1)^m G(t) \to \infty \quad \text{as} \quad t \to -\infty\,, \\ \text{ii)} & 2\dot{G}/\dot{F} = -1 + 2\dot{E}/\dot{F} \quad \text{and} \quad (-1)^m \dot{F} < 0 \quad \text{on} \quad (-\infty, 0)\,. \end{array} \tag{22.1}$$

If m is even, Remark 11.2 and (22.1.ii) give $2\dot{G}/\dot{F} > 1$, and hence $2\dot{G} < \dot{F} < 0$, on $(-\infty, 0)$. As $G(0) = E(0) < 0$ by (5.6), this and (22.1.i) show that z exists, is unique, and (21.4) holds.

Now let m be odd. By (22.1.ii) and Remark 11.2, $2\dot{G}/\dot{F}$ decreases on $(-\infty, 0)$ from 1 to $-\infty$, assuming the value 0 at a unique $w < 0$. Since $\dot{F} > 0$ on $(-\infty, 0)$ (see (22.1.ii)), we have $\dot{G} > 0$ on $(-\infty, w)$ and $\dot{G} < 0$ on $(w, 0)$. Thus $G(w)$ is the maximum value of G in $(-\infty, 0]$, and so $G(w) > G(0) = E(0)$. As G increases on $(-\infty, w)$ from $-\infty$ (see (22.1.i)) to $G(w)$, it assumes the intermediate value $E(0)$ exactly once in $(-\infty, w)$, and not at all in $[w, 0)$ (where it decreases to the limit $E(0)$), which gives the existence and uniqueness of z and the relation $z < w$. Finally, on $(-\infty, 0)$ we have $F < 0$ and $E/F > 1/2$ by (6.1), and hence $G = E - F/2 < 0$, so that $G(w) < 0$.

23. Positivity and type (17.6.b)

In §20 we classified all those pairs Q, I with (2.5.b) and (2.5.e) that are of type (17.6.b). We will now determine which of them also satisfy the positivity condition of §16 or, equivalently, all the remaining conditions in (2.5).

Proposition 23.1. *Given $u \in \mathbf{R} \smallsetminus \{0, 1, 2\}$, let $v = u^*$ with $u^* = u/(u-1)$, cf.* (20.2). *Also, let $\mathbf{V}$ be the space* (2.4) *for a fixed integer $m \geqslant 2$, and let $Q \in \mathbf{V} \smallsetminus \{0\}$ be a function with* (2.5.b) *on the closed interval I with the endpoints u, v. By* Lemma 9.1, *such Q exists and is unique up to a factor. Then, the pair Q, I satisfies all five conditions listed in* (2.5) *if and only if*

a) *m is even, or*

b) *m is odd, while $u \notin [z, w]$ and $v \notin [z, w]$, with z, w as in* §21.

Proof. Relation $v = u^*$ amounts to (17.6.b), and hence implies (2.5.e) (Theorem 17.5). Therefore we just need to show that (a) or (b) holds if and only if Q, I satisfy (2.5.a), (2.5.c), (2.5.d), that is, the positivity condition of §16.

First, let one of u, v lie in $(1,2) \cup (2,\infty)$. Then so does the other (Lemma 20.1), and the positivity condition for Q, I follows from (i) in §16, while, at the same time, we have (a) or (b), as $[z,w] \subset (-\infty,0)$ when m is odd (see §21).

Thus, it suffices to prove our assertion in the remaining case where u and $v = u^*$ both lie in $(-\infty,0) \cup (0,1)$, that is (cf. Lemma 20.1), one of them is in $(-\infty,0)$ and the other in $(0,1)$. The hypotheses of Proposition 15.2 now are satisfied; also, $F(u) \neq 0$ by (2.1), so that (21.3) gives $B \neq 0$, $C/B = -1/2$ and $A/B = -G(u)$. Using (21.3), we may now rephrase conditions (i), (ii) of Proposition 15.2 as follows:

i) $\dot G(u)/\dot F(u) < 0 < \dot G(v)/\dot F(v)$, provided that u and $v = u^*$ have been switched, if necessary, so that $(-1)^m(u-v) > 0$.

ii) $G(u) < E(0)$.

We will show that (a) or (b) holds if and only if one of these conditions (i), (ii) is satisfied, thus obtaining our assertion as a direct consequence of Proposition 15.2.

Let us first suppose that neither (a) nor (b) holds, that is, m is odd and one of u, v lies in $[z,w]$. As $w < 0$, ordering u, v as in (i) (so that $u < v$), we obtain $u \in [z,w]$ and $v \in (0,1)$. Since $G(u) \geqslant E(0)$ (see (21.5)), condition (ii) cannot be satisfied. Also, $\dot G(u) \geqslant 0$ (see (21.5)); as $\dot F > 0$ on $(-\infty,0)$ for odd m by (5.1), this gives $\dot G(u)/\dot F(u) \geqslant 0$, and (i) fails as well. Thus, if (i) or (ii) is satisfied, we must have (a) or (b).

To prove the converse, let us now assume (a) or (b) and order u, v as in (i). If m is even, we thus have $u^* = v < 0 < u < 1$ and $\dot G(v) < 0$ (cf. (21.4)). Hence, as (5.1) yields $\dot F < 0$ on $(-\infty,0) \cup (0,1)$, (21.2.b) implies the double inequality in (i). On the other hand, if m is odd, we have $u < 0 < v < 1$ and, as $w < 0$, condition (b) leads to two possible cases: $u < z$, or $w < u < 0$. If $u < z$, (21.5) gives $G(u) < E(0)$, that is, (ii); while, if $w < u < 0$, (21.5) shows that $\dot G(u) < 0$, which, combined with (21.2.b) and the fact that $\dot F > 0$ on $(-\infty,0)$ by (5.1), again gives the double inequality in (i). This completes the proof. $\square$

24. More on the polynomial T

Given an integer $m \geqslant 2$, let α, β, ϕ, f be the polynomials in u, v with

$$\begin{aligned}
&\text{a)}\quad \alpha = (u-1)^m(v-2)v^{2m-1}E(u) - (v-1)^m(u-2)u^{2m-1}E(v),\\
&\text{b)}\quad \beta = (v-1)^m(u-2)u^{2m-1} - (u-1)^m(v-2)v^{2m-1},\\
&\text{c)}\quad \phi = u+v-2, \qquad f = muv-(2m-1)\phi.
\end{aligned} \tag{24.1}$$

for E as in (2.1). Thus, with A, B depending on $u, v \in \mathbf{R} \smallsetminus \{1\}$ as in (10.3),

$$\alpha = (u-1)^m(v-1)^m A, \qquad \beta = (u-1)^m(v-1)^m B. \tag{24.2}$$

Lemmas 17.3 and 17.2 imply that the product of $u(u-2)v(v-2)$ and the right-hand side of (17.5) equals $(u-1)^m(v-1)^m$ times $2(uv-u-v)$ times the expression (17.3). Dividing by $2(uv-u-v)$ and using (17.5), (24.1), (24.2), we obtain, for the polynomial T given by (17.5),

$$\begin{aligned}
&\text{i)}\quad (v-u)^3u(u-2)v(v-2)T(u,v) = f\alpha + c\beta\phi, \qquad \text{where}\\
&\text{ii)}\quad c = (2m-1)\Sigma(0) > 0, \qquad \text{with} \quad \Sigma(0) \quad \text{as in (5.6.i).}
\end{aligned} \tag{24.3}$$

(This c is not related to the symbol c in §0 – §4.) For $v = 1$, (24.1) gives $\alpha = -(u-1)^m E(u)$, $\beta = (u-1)^m$ and $\phi = u-1$, $f = (1-m)u + 2m - 1$, so that (24.3.i) with $v = 1$ becomes

$$\begin{aligned}
&\text{a)}\quad u(u-2)T(u,1) = (u-1)^{m-2}W(u) \qquad \text{for all} \quad u \in \mathbf{R}, \quad \text{where}\\
&\text{b)}\quad W(u) = [(m-1)u-2m+1]\Sigma(u) + (2m-1)\Sigma(0) \quad \text{for} \quad u \in \mathbf{R}.
\end{aligned} \tag{24.4}$$

Also, with F, Σ, G, T given by (2.1), (2.2), (21.1), (17.5) for an integer $m \geqslant 2$,

$$\begin{array}{ll} \text{a)} & u^5(u-2)^5\,T(u,u^*) \,=\, 2(u-1)^2 F(u)S(u) \quad \text{for all } u \neq 1, \quad \text{where} \\ \text{b)} & S(u) = [(m-1)u^2 - 2(2m-1)(u-1)]\,G(u) + (2m-1)(u^2-2u+2)\Sigma(0) \end{array} \tag{24.5}$$

and $u^* = u/(u-1)$ for $u \neq 1$. In fact, setting $v = u^*$ in (24.1.c) we get $(u-1)\phi = (u-1)^2 + 1$ and $(1-u)f = (m-1)u^2 - 2(2m-1)(u-1)$. Since $(u-1)(v-u) = (u-1)^2 v(v-2) = u(2-u)$ whenever $v = u^*$, we obtain (24.5.a) by setting $v = u^*$ in (24.3.i) and noting that (24.2) with $v = u^*$ reads $\alpha = A$ and $\beta = B$ (cf. (20.2.b)), while A, B with $v = u^*$ are in turn given by (21.3).

Finally, for any integer $m \geqslant 2$, applying $\partial/\partial u$ to (24.3.i) at $u = 0$ and using (24.1) with $\dot{E}(0) = 0$ (see (5.6.iii)), we obtain

$$2T(0,v) = (-1)^m m\Sigma(0)v^{2m-4} \quad \text{for all } v \in \mathbf{R}, \text{ with } \Sigma(0) \text{ as in (5.6.i).} \tag{24.6}$$

25. The sign of T on a specific curve

Let T be the polynomial in u, v given by (17.5) for any fixed integer $m \geqslant 2$. In this section we describe the behavior of $\operatorname{sgn} T$ on the intersection of the half-plane $u < 0$ with the hyperbola $\mathcal{H}$ given by $uv = u + v$ (that is, $v = u^*$, cf. (20.2)). This will allow us, later in §29, to draw important conclusions about zeros of T in the region $u < 0 < v \leqslant 1$.

We need the next result to prove Lemmas 25.2 and 26.1. See also Example 27.4.

Lemma 25.1. *Given $c \in \mathbf{R} \smallsetminus \{0\}$ and C^1 functions $\alpha, f, \psi, \lambda, \mu, \Psi$ on an open interval, let $(\;)'$ be the derivative operator followed by multiplication by some fixed C^1 function. If $\alpha' = \lambda\alpha - c\mu$ and $\Psi = f\alpha + c\psi$, then, at every point where $\Psi = 0$,*

a) $c^{-1}f\Psi' = (\psi' - \lambda\psi - f\mu)f - \psi f'$,

b) $c^{-1}f\Psi' = (f\phi' - \phi f')\beta - (\phi\nu + f\mu)f$ *provided that, in addition, $\psi = \beta\phi$ and $\beta' = \beta\lambda - \nu$ for some C^1 functions ϕ, β, ν.*

In fact, any point with $\Psi = 0$ has $f\alpha = -c\psi$, and hence $f\alpha' = \lambda f\alpha - cf\mu = -c(\lambda\psi + f\mu)$, so that (a) is immediate as $\Psi = f\alpha + c\psi$ and $f(f\alpha)' = (f\alpha')f + (f\alpha)f'$. To obtain (b), it now suffices to replace ψ and ψ' in (a) by $\beta\phi$ and, respectively, $\beta\phi' + \beta'\phi = \phi'\beta + (\beta\lambda - \nu)\phi$.

Lemma 25.2. *Given an integer $m \geqslant 2$, let T, S be the polynomial and rational function with* (17.5) *and* (24.5.b)*. Also, let $u^* = u/(u-1)$ for $u \neq 1$, cf.* (20.2).

a) *If m is even, $T(u,u^*) > 0$ for every $u < 0$.*

b) *If m is odd, equation $T(x,x^*) = 0$ has a unique negative real solution x. With this x we have $T(u,u^*) > 0$ for all $u \in (-\infty, x)$ and $T(u,u^*) < 0$ for all $u \in (x, 0)$, as well as $x < z$, where z is defined as in* §21.

c) *If m is odd, $\operatorname{sgn}(u - x) = \operatorname{sgn} S(u)$ for x as in* (b), *sgn as in* §4, *and any $u \in (-\infty, 0]$.*

d) *$d[T(u,u^*)]/du < 0$ at every $u < 0$ with $T(u,u^*) = 0$.*

Proof. By (21.2.d), $S(0) = \dot{S}(0) = 0$ and $(2m-3)\ddot{S}(0) = 8m(m-1)\Sigma(0) > 0$, where $(\;)^{\cdot} = d/dt$. Thus, $S(t) > 0$ for $t < 0$ near 0. Also, $(-1)^m S(t) \to \infty$ as $t \to -\infty$, since, by (22.1.i), $(-1)^m S(t)/t^2 \to \infty$ as $t \to -\infty$.

In view of (21.2.c), the assumptions of Lemma 25.1 are satisfied by $\alpha = G(t)$ with (21.1), $f = (m-1)t^2 - 2(2m-1)(t-1)$, $\psi = t^2 - 2t + 2$, $\lambda = \Lambda(t)$ with (5.1.b), $\mu = 2(t-1)$, $\Psi = S(t)$, the constant (24.3.ii), and $(\;)' = t(t-1)(t-2)\,d/dt$. Also, $f = \lambda - t^2$ (by (5.1.b)) and $\psi = t^2 - \mu$, so that $\lambda\psi + f\mu = \lambda t^2 - \lambda\mu + \lambda\mu - \mu t^2 = (\lambda - \mu)t^2 = mt^2(t-2)^2$ and $\psi' - \lambda\psi - f\mu = -t(t-2)[(m-2)t(t-2) - 2]$. Moreover, $\psi f' = 2t(t-1)(t-2)(t^2 - 2t + 2)[(m-1)t - 2m + 1]$. The right-hand side in Lemma 25.1(a)

thus equals $-m(m-1)t^2(t-2)^4$. Using Lemma 25.1(a), with both sides divided by $t(t-2)$, we see that $(t-1)f\dot{S}(t) = -m(m-1)ct(t-2)^3$ at every $t \in \mathbf{R} \smallsetminus \{0,1,2\}$ at which $S(t)=0$. This gives $\dot{S}(t) > 0$ for every $t < 0$ with $S(t) = 0$, as the quadratic polynomial f, having a positive value and a negative derivative at $t = 0$, must be positive for all $t < 0$. Remark 18.1 for $\Phi = S$ and $\mathcal{I} = (-\infty, 0)$, combined with the signs of S near $-\infty$ and 0 determined in the last paragraph, now gives (c) with some (unique) $x < 0$, as well as $S > 0$ on $(-\infty, 0)$ if m is even. Since $S(x) = 0$, the definition of S leads to a rational expression for $G(x)$ in terms of x, which easily shows that $G(x) < E(0) = -\varSigma(0)$ (cf. (5.6.ii)), so that $x < z$ by (21.5). Also, as $\operatorname{sgn} F = (-1)^m$ on $(-\infty, 0)$, cf. (6.1), relation (24.5.a) yields $\operatorname{sgn} T(u,u^*) = (-1)^m \operatorname{sgn} S(u)$ for $u < 0$, with sgn as in §4. Hence (a), (b) and (d) follow, completing the proof. ☐

26. The differential of T at points where $T = 0$

Lemma 26.1. *Given an integer $m \geqslant 2$, let $T, \beta, \phi, f, \varPsi, \zeta, \vartheta$ be the polynomials in u,v given, respectively, by* (17.5), (24.1.b), (24.1.c), *formula*

$$\varPsi \,=\, (v-u)^3 u(u-2)v(v-2)T(u,v)\,, \tag{26.1}$$

and $\zeta = (u-1)^{m-1}(v-2)v^{2m-1}$, $\vartheta = (v-1)^{m-1}(u-2)u^{2m-1}$. For the constant c with (24.3.ii) *we then have, at every point (u,v) at which $\varPsi = 0$,*

$$\begin{aligned}(mc)^{-1}f\,\partial\varPsi/\partial u \;&=\; (u-v)f\zeta \,-\, v(v-2)\beta \quad \text{unless} \quad u(u-2)=0\,,\\ (mc)^{-1}f\,\partial\varPsi/\partial v \;&=\; (u-v)f\vartheta \,-\, u(u-2)\beta \quad \text{unless} \quad v(v-2)=0\,.\end{aligned} \tag{26.2}$$

Proof. The assumptions of Lemma 25.1(b) hold for $\beta, \phi, f, \varPsi, c$ chosen as above, α as in (24.1.a), $\psi = \beta\phi$, $\lambda = 2(mu - 2m + 1)$, $\mu = 2(u-1)\zeta$, $\nu = -\varLambda(u)\zeta$, with $\varLambda$ given by (5.1.b), and $(\;)' = u(u-2)\,\partial/\partial u$, where the variable v is fixed.

This is clear as $\varPsi = f\alpha + c\phi\beta$ by (26.1) and (24.3.i), while $\alpha' = \lambda\alpha - c\mu$ and $\beta' = \beta\lambda - \nu$ in view of (5.5.ii) with $\varLambda(u) + mu(u-2) = 2(u-1)(mu-2m+1)$, which is immediate from (5.1.b).

Now $\phi' = u(u-2)$, $f' = u(u-2)l(v)$, and so $f\phi' - \phi f'$ equals $u(u-2)$ times $f - \phi l(v) = muv - [2m-1+l(v)]\phi = mv(u-\phi) = -mv(v-2)$ with $l(v) = mv - 2m + 1$, that is, $f\phi' - \phi f' = -muv(u-2)(v-2)$.

Finally, $(2m-1)\mu - \nu = [2(2m-1)(u-1) + \varLambda(u)]\zeta = mu^2\zeta$ (cf. (5.1.b)). Therefore, $\phi\nu + f\mu = \phi\nu + [muv - (2m-1)\phi]\mu = -[(2m-1)\mu - \nu]\,\phi + muv\mu = -mu^2\phi\zeta + muv\mu = -mu(u-2)(u-v)\zeta$. Dividing both sides of the equality in Lemma 25.1(b) by $mu(u-2)$, we arrive at the first relation in (26.2). The second one is obtained by evaluating the first at the point (v,u) rather than (u,v). In fact, switching u,v causes ζ to be replaced by ϑ, while f is symmetric in u,v and $\beta, \varPsi$ are anti-symmetric, so that the values of $\partial\varPsi/\partial u$ at (v,u) and $\partial\varPsi/\partial v$ at (u,v) are mutually opposite. This completes the proof. ☐

Lemma 26.2. *For any integer $m \geqslant 2$, let $R, \tilde{\varXi}$ be the polynomials in real variables u,v with $R=1$ if m is even, $R = uv(u+v) - (u^2+v^2)$ if m is odd, and $\tilde{\varXi} = [(u-1)v^2]^{m-1} - [(v-1)u^2]^{m-1}$ for all m. Then $\tilde{\varXi} = (v-u)(uv-u-v)R\varXi$ for some polynomial $\varXi$ in u,v such that $\varXi > 0$ on $\mathbf{R}^2 \smallsetminus \{(0,0),(1,1)\}$.*

This is clear since $\xi^{2n} - \eta^{2n} = (\xi-\eta)(\xi+\eta)(\xi^{2n-2} + \xi^{2n-4}\eta^2 + \ldots + \eta^{2n-2})$ and $\xi^{m-1} - \eta^{m-1} = (\xi-\eta)(\xi^{m-2} + \xi^{m-3}\eta + \ldots + \eta^{m-2})$ for an integer $n \geqslant 1$ or an even integer $m \geqslant 2$, and $\xi, \eta \in \mathbf{R}$. In both cases, the last factor is positive unless $\xi = \eta = 0$. (In fact, if m is even and $\eta \neq 0$, then η is both a simple root and the unique real root of the polynomial $\xi \mapsto \xi^{m-1} - \eta^{m-1}$.) Our assertion now follows if we use $n = (m-1)/2$ when m is odd, and set $\xi = (u-1)v^2$, $\eta = (v-1)u^2$, so that $\xi - \eta = (v-u)(uv-u-v)$ and $\xi + \eta = uv(u+v) - (u^2+v^2)$.

For f, R depending on (u,v) as in (24.1.c) and Lemma 26.2 with a fixed $m \geqslant 2$,

$$f \;\geqslant\; m \;>\; 0 \quad \text{and} \quad (-1)^m R \;\geqslant\; 0 \quad \text{whenever} \quad u-1 \leqslant 0 < v \leqslant 1. \tag{26.3}$$

In fact, $f = m\tilde{u}\tilde{v} + (m-1)(\tilde{u}+\tilde{v}) + m$ for $\tilde{u} = 1-u$ and $\tilde{v} = 1-v$, so that $f \geqslant m$ as $\tilde{u}, \tilde{v} \geqslant 0$. That $(-1)^m R \geqslant 0$ follows if one adds up the inequalities obtained from multiplying the relations $v \leqslant 1$ and $u \geqslant 1$ by u^2 and, respectively, v^2.

Lemma 26.3. *Given an integer $m \geqslant 3$, for* sgn *and T as in* §4 *and* Lemma 17.3 *we have, at every point $(u,v) \in \mathbf{R}^2$ at which $u < 0 < v \leqslant 1$ and $T(u,v) = 0$,*

$$(-1)^m \operatorname{sgn} d_{\mathbf{w}} T \;=\; \operatorname{sgn}(uv - u - v) \qquad \text{and} \qquad dT \;\neq\; 0\,, \tag{26.4}$$

$d_{\mathbf{w}}$ being the directional derivative for the vector field $\mathbf{w} = (u-1, v-1)$ on $\mathbf{R}^2$.

Proof. With $c, f, \varPsi$ as in (24.3.ii), (24.1.c) and (26.1), let $\varPhi$ stand for the value of $(mc)^{-1} f\, d_{\mathbf{w}}\varPsi$ at any fixed (u,v) with $u < 0 < v \leqslant 1$ and $T(u,v) = 0$. Thus, $d_{\mathbf{w}}$ applied to (26.1) gives $\operatorname{sgn} d_{\mathbf{w}} T = -\operatorname{sgn} \varPhi$ at (u,v), as $cf > 0$ (see (26.3). Also, $f = (u-2)(mv-2m+1)+v$. Hence $\varPhi$, being, by (26.2), the difference of $(u-v)[(u-1)^m(v-2)v^{2m-1} + (v-1)^m(u-2)u^{2m-1}]f$ and $u(u-2)(v-1)\beta + v(v-2)(u-1)\beta$ at (u,v), must equal the sum of the two expressions

$$\begin{aligned}
&(u-v)\,[(u-2)(mv-2m+1)+v]\,[(u-1)^m(v-2)v^{2m-1} + (v-1)^m(u-2)u^{2m-1}],\\
&[u(u-2)(v-1)+v(v-2)(u-1)]\,[(u-1)^m(v-2)v^{2m-1} - (v-1)^m(u-2)u^{2m-1}].
\end{aligned}$$

In both displayed products, either of the two factors enclosed in square brackets is a sum of two polynomials; of these eight polynomials, four are manifestly divisible by $u-2$. Direct multiplications in both displayed lines, performed to evaluate $\varPhi$, thus give rise to eight product terms, of which only two, $v(v-2)(u-1)$ times $(u-1)^m(v-2)v^{2m-1}$ and $(u-v)v$ times $(u-1)^m(v-2)v^{2m-1}$, fail to explicitly contain $u-2$ as a factor; their sum, however, is $(u-2)(v-1)(u-1)^m(v-2)v^{2m}$. Thus, the polynomial in u, v representing the value $\varPhi$ is divisible by $u-2$, and the calculation just outlined gives

$$\begin{aligned}
\varPhi/(u-2) \;\;=\;\; & u(u-1)^m(v-1)(v-2)v^{2m-1}\\
- \;\; & (uv-u-v)\,[u+(v-2)]\,u^{2m-1}(v-1)^m\\
+ \;\; & (u-v)\,[(mu-2m+1)(v-2) \;+\; u]\,u^{2m-1}(v-1)^m\\
+ \;\; & (v-2)\,[(mv-2m+1)u - (m-1)(v-2)v]\,(u-1)^m v^{2m-1}.
\end{aligned} \tag{26.5}$$

(The second line of (26.5) is the result of combining two of the four terms containing the factor $u^{2m-1}(v-1)^m$ with the aid of (17.4).) The right-hand side can, as before, be rewritten as the sum of polynomial product terms, with each of the three square brackets contributing to two of them; this time, there are *seven* such terms, and five of them are manifestly divisible by $v-2$, while the other two add up to $[(u+v-uv)+(u-v)]\,u^{2m}(v-1)^m$, that is, $v-2$ times $-u^{2m+1}(v-1)^m$. Consequently, as a polynomial in u, v, our $\varPhi/(u-2)$ is divisible by $v-2$ and, proceeding just as we did above to evaluate $\varPhi/(u-2)$, we obtain

$$\begin{aligned}
\varPhi/[(u-2)(v-2)] \;\;=\;\; & [(m-1)u^2 - (m+1)uv - 2(m-1)u + 2mv]\,u^{2m-1}(v-1)^m\\
- \;\; & [(m-1)v^2 - (m+1)uv - 2(m-1)v + 2mu]\,v^{2m-1}(u-1)^m.
\end{aligned} \tag{26.6}$$

Relation $[(v-1)u^2]^{m-1} = [(u-1)v^2]^{m-1} - \widetilde{\varXi}$, for $\widetilde{\varXi}$ as in Lemma 26.2, allows us to replace the factor $u^{2m-1}(v-1)^m = [(v-1)u^2]^{m-1}(v-1)u$ with $-(v-1)u\widetilde{\varXi} + (v-1)u(u-1)^{m-1}v^{2m-2}$, while, $\widetilde{\varXi} = (v-u)(uv-u-v)R\varXi$ by Lemma 26.2. It now easily follows that $\varPhi$ is equal to $(u-2)(v-2)(u-v)(uv-u-v)$ times

$$\begin{aligned}
&[(m-1)u^2 \;-\; (m+1)uv \;-\; 2(m-1)u \;+\; 2mv]\,(v-1)\,uR\varXi\\
&\qquad\qquad\qquad\qquad + \;(m-1)(u+v-2)(u-1)^{m-1}v^{2m-2}.
\end{aligned} \tag{26.7}$$

As $u < 0 < v \leqslant 1$, all four terms within the square brackets in (26.7) are positive. The signs of the other four factors $(v-1), u, R, \varXi$ in that line are, respectively, ($-$ or 0), $-$, ($(-1)^m$ or 0), $+$, cf.

(26.3) and Lemma 26.2, while those of the four factors in the second line are $+, -, (-1)^{m-1}, +$, so that sgn of (26.7) equals $(-1)^m$, which clearly yields the first relation in (26.4). The second one then is immediate from the first, since, by Lemma 25.2(d), $dT \neq 0$ at our (u,v) also in the case where $v = u^*$, that is, $uv - u - v = 0$. This completes the proof. □

Remark 26.4. By (26.4), 0 is a regular value of T in $\mathcal{R} = (-\infty, 0) \times (0, 1]$, that is, in the region $\mathcal{R}$ in the uv-plane $\mathbf{R}^2$ given by $u < 0 < v \leqslant 1$. Thus, if the set of zeros of T in $\mathcal{R}$ is nonempty, its connected components are one-dimensional real-analytic submanifolds of $\mathcal{R}$, possibly with boundary. When m is odd, that set of zeros is nonempty, as it contains (x, x^*) (see Lemma 25.2(b)).

27. An analytic curve segment with $T = 0$

The meaning of the symbol r in this section is not related to its use in §3.

Lemma 27.1. *Let a real number $q_* \in [-1, 0)$ and a C^1 function Y of the real variables q, r, defined on the square $\mathcal{S} = (-1, 0) \times [0, 1]$, satisfy the conditions*

i) $\partial Y/\partial r < 0$ *at every interior point of* $\mathcal{S}$ *at which* $Y = 0$,

ii) $\operatorname{sgn} Y(q, 1) = \operatorname{sgn}(q - q_*)$ *and* $Y(q, 0) > 0$

for all $q \in (-1, 0)$, *with* sgn *as in* §4. *Then*

a) $Y > 0$ *on* $(q_*, 0) \times [0, 1]$.

b) *For every* $q \in (-1, q_*]$ *there exists a unique* $r \in [0, 1]$ *with* $Y(q, r) = 0$.

c) *The coordinate function* q, *restricted to the set of all zeros of* Y *in the square* $\mathcal{S}$, *maps it homeomorphically onto* $(-1, q_*]$. *In particular, that set of zeros is empty when* $q_* = -1$.

In fact, by (ii), both $Y(q, 0)$ and $Y(q, 1)$ are positive if $q \in (q_*, 0)$, so that (i) and Remark 18.1 with $\Phi(r) = -Y(q, r)$ and $\mathcal{I} = [0, 1]$ yield (a). Next, r required in (b) exists for $q \in (-1, q_*]$, since (ii) gives $Y(q, 0) > 0 \geqslant Y(q, 1)$, and it is unique in view of Remark 18.1 (for Φ as above) and (i). Finally, q sends the set of zeros of Y bijectively onto $(-1, q_*]$, while continuity of its inverse mapping (that is, of the function $q \mapsto r$) follows from an obvious subsequence argument.

We define a rational function q of the variables u, v and a constant q_* by

$$q = (v - u)/(u + v - 2), \qquad q_* = (2 - x)x/[(x - 1)^2 + 1] \in (-1, 0), \tag{27.1}$$

for x as in Lemma 25.2(b). Thus, x and q_* also depend on an odd integer $m \geqslant 3$.

Remark 27.2. We have $q_* \in (-1, 0)$ in (27.1) since the assignment $x \mapsto q_*$ is an increasing diffeomorphism $(-\infty, 0) \to (-1, 0)$. In fact, it is the composite $(-\infty, 0) \to (1, \infty) \to (-1, 0)$ of the decreasing diffeomorphisms $x \mapsto \zeta = (x - 1)^2$ and $\zeta \mapsto q_* = (1 - \zeta)/(1 + \zeta) = -1 + 2/(1 + \zeta)$.

Theorem 27.3. *Let Z be the set of all $(u, v) \in \mathbf{R}^2$ with $u < 0 < u^* \leqslant v \leqslant 1$ and $T(u, v) = 0$, where $u^* = u/(u - 1)$ and T is the polynomial with* (17.5) *for a given integer $m \geqslant 2$. Also, let q, q_*, x be as in* (27.1) *and* Lemma 25.2(b).

a) *If m is even, Z is empty.*

b) *If m is odd, Z is a real-analytic compact curve segment embedded in $\mathbf{R}^2$.*

For odd m, there exists a unique negative real number y with $T(y, 1) = 0$. The endpoints of the curve segment Z then are (x, x^*) and $(y, 1)$, while the restriction of q to Z is a homeomorphism $q : Z \to [-1, q_*]$ sending $(y, 1)$ and (x, x^*) onto -1 and, respectively, q_*.

Proof. At any (u, v) with $u < 0$ and $v = 1$, (26.4) reads: $(-1)^m d[T(u, 1)]/du > 0$, provided that $T(u, 1) = 0$. The assumptions of Remark 18.1 thus are satisfied by $\Phi(u) = (-1)^m T(u, 1)$ on the

interval $\mathcal{I} = (-\infty, 0)$. Also, $\Phi(0) > 0$ by (24.6), while $(-1)^m\Phi(u) \to \infty$ as $u \to -\infty$ (since W in (24.4) is a degree m polynomial with leading coefficient $m-1 > 0$, cf. (2.2)). Remark 18.1 now shows that equation $T(y,1) = 0$ has no negative real solutions y when m is even, and has exactly one such solution when m is odd.

Formula $\Upsilon(u,v) = (q,r)$ with q as in (27.1) and $r = u(v-1)/v$ defines a diffeomorphism $\Upsilon : \mathcal{K} \to \mathcal{S}$ of the set $\mathcal{K}$ in the uv-plane, formed by all (u,v) with $u \leqslant 0 < v < 1$ and $u^* \leqslant v$, onto the square $\mathcal{S} = (-1,0) \times [0,1]$ in the qr-plane. This is an easy exercise; for instance, $u+v-2 < 0$ on $\mathcal{K}$, as $u \leqslant 0$ and $v < 2$, so that $-1 < q < 0$, while $r = u/v^*$, and so $0 \leqslant r \leqslant 1$ as $v^* \leqslant u \leqslant 0$, that is, $0 \leqslant u^* \leqslant v$. (Cf. Lemma 20.1.) Also, solving $v - u = (u+v-2)q$ for v, we can rewrite condition $r = u(v-1)/v$, for any $(q,r) \in \mathcal{S}$, as the equation $(q+1)(u-r-1)u + 2qr = 0$, which is quadratic in u and has a positive leading coefficient, while its left-hand side is nonpositive at both $u = 0$ and $u = 1$, so that its only nonpositive real root is simple. Solving it for u and using our expression for v in terms of u, q, we now get an explicit description of the inverse Υ^{-1}.

The diffeomorphism $\Upsilon : \mathcal{K} \to \mathcal{S}$ sends $(-1)^m T : \mathcal{K} \to \mathbf{R}$ and the vector field $\mathbf{w} = (u-1, v-1)$ on $\mathcal{K}$ onto the function $Y = (-1)^m T \circ \Upsilon^{-1} : \mathcal{S} \to \mathbf{R}$ and a vector field on $\mathcal{S}$. The latter equals a positive function times the coordinate vector field in the direction of r, which is clear since $d_{\mathbf{w}} q = 0$ for q treated as a function of u, v, while $v^2 d_{\mathbf{w}} r = (v-1)[(u-1)v + u] > 0$ on $\mathcal{K}$, as $v > 0$ and $v-1$, $u-1$, u are all negative. Thus, by (26.4), our Y satisfies (i) in Lemma 27.1, since the inequality $uv \leqslant u + v$ (that is, $u^* \leqslant v$) gives $uv - u - v < 0$ on the interior of $\mathcal{K}$.

Moreover, Υ also maps the boundary curves of $\mathcal{K}$, parameterized by $u \mapsto (u, u^*)$ and, respectively, $v \mapsto (0, v)$ with $u \in (-\infty, 0)$ and $v \in (0,1)$, onto the boundary curves for $\mathcal{S}$, given by $q \mapsto (q,1)$ and $q \mapsto (q,0)$ with $q \in (0,1)$, in such a way that the curve parameter q is an increasing (or, respectively, decreasing) function of u (or, v). Lemma 25.2(a),(b) and (24.5) now show that Y satisfies condition (ii) in Lemma 27.1 as well, provided that we set $q_* = -1$ when m is even, and define q_* as in (27.1) when m is odd. Note that, for odd m, the function on $\mathcal{K}$ corresponding under Υ to the coordinate function q on $\mathcal{S}$ is, obviously, $q : \mathcal{K} \to \mathbf{R}$ given by (27.1), the value of which at (x, x^*) is q_*.

Lemma 27.1 and Remark 26.4, combined with our initial conclusion about the equation $T(y,1) = 0$, now yield both (a) and (b), completing the proof. $\square$

Example 27.4. If $m = 3$ we have $x = -(\sqrt{5}+1)/2$ and $y = -3/2$ for x, y as in Lemma 25.2(b) and Theorem 27.3. This is clear from Lemma 25.2(c) and (24.4) since, using (2.1) and the explicit expressions for Σ and E in Example 19.4, we obtain $(t-1)^3 S(t) = t^2(t-2)^4(t^2+t-1)$ and $W(t) = t(t-2)(2t+3)$. Now $x^2 = 1 - x$, so that $(x-1)^2 + 1 = 3(1-x)$ and hence $q_* = -\sqrt{5}/3$.

28. A further symmetry

In view of Lemma 20.1, for u^*, v^* as in (20.2) the assignment

$$\mathcal{U} \ni (u,v) \;\mapsto\; (v^*, u^*) \in \mathcal{U}\,, \quad \text{where} \quad \mathcal{U} \,=\, (\mathbf{R} \smallsetminus \{1\}) \times (\mathbf{R} \smallsetminus \{1\})\,, \tag{28.1}$$

is an involution $\mathcal{U} \to \mathcal{U}$ and its fixed-point set is the hyperbola $\mathcal{H}$ given by $v = u^*$. The importance of (28.1) for our discussion is due to Proposition 28.2 below.

Both $\mathcal{H}$ and (28.1) appear particularly simple if one replaces u, v by new affine coordinates a, b with $u = a+1$, $v = b+1$. The equation of $\mathcal{H}$ then becomes $ab = 1$ (cf. (20.2.b)), while (28.1) reads $(a,b) \mapsto (1/b, 1/a)$. In other words, let us shift the origin from $(u,v) = (0,0)$ to $(u,v) = (1,1)$, so as to treat the original uv-plane as the ab-plane $\mathbf{R}^2$ with linear coordinates a, b. We may now endow the latter with the indefinite inner product corresponding to the quadratic function $(a,b) \mapsto ab$. This makes $\mathcal{H}$ a unit pseudocircle centered at the origin, while (28.1) acts through division of any

non-null vector (a,b) by its inner square ab. Thus, (28.1) forms the pseudo-Euclidean analogue of the Euclidean conformal inversion $\mathbf{x} \mapsto \mathbf{x}/|\mathbf{x}|^2$.

Lemma 28.1. *For F, E, Π as in (2.1) and (17.2) with a fixed integer $m \geqslant 2$, let $\tilde F(t) = (t-1)\dot F(t)$, $\tilde E(t) = (t-1)\dot E(t)$, where $(\ )^{\boldsymbol\cdot} = d/dt$. We then have $\tilde F(t^*) = \tilde F(t)$, $\tilde E(t^*) = \tilde F(t) - \tilde E(t)$, and $\Pi(u^*,v^*) = \Pi(u,v)$ for any $t,u,v \in \mathbf{R} \smallsetminus \{1\}$, with t^*, u^*, v^* defined as in (20.2).*

In fact, $dt^*/dt = -1/(t-1)^2$ by (20.2.b). Applying d/dt and the chain rule to (20.1), we get $\dot F(t^*) = (t-1)^2 \dot F(t)$ and $\dot E(t^*) = (t-1)^2[\dot F(t) - \dot E(t)]$, which yields the first two relations, and (with (17.2), (20.1) and (20.2.b)), also the third.

Proposition 28.2. *Given an integer $m \geqslant 2$, let $T(u,v)$ be as in (17.5), and let $u,v \in \mathbf{R} \smallsetminus \{1\}$. Then, with t^*, u^*, v^* defined as in (20.2), and sgn as in §4,*

- a) *$\widetilde T(u^*,v^*) = \widetilde T(u,v)$, where $\widetilde T(u,v) = (u-1)^{2-m}(v-1)^{2-m}T(u,v)$.*
- b) *The involution (28.1) leaves invariant the function $\operatorname{sgn} T$.*
- c) *The set $\{(u,v) \in \mathbf{R}^2 \mid T(u,v) = 0$ and $u \neq 1 \neq v\}$ is invariant under (28.1).*

This is clear as $(u-1)(v-1)\theta(u^*,v^*) = -\theta(u,v)$ both for $\theta(u,v) = v-u$ and for $\theta(u,v) = uv-u-v$. Using (17.5), Lemma 28.1 and (20.2.b), we now obtain $(u-1)^{2m-4}(v-1)^{2m-4}\,T(u^*,v^*) = T(u,v)$, and (20.2) gives (a), while (a) with $T(u,v) = T(v,u)$ yields (b), and (b) implies (c).

Remark 28.3. The involution (28.1) admits an interesting algebraic-geometric interpretation in terms of the projective plane in which the ab-plane corresponding (as described above) to the original uv-plane is canonically embedded. Namely, the homogeneous-coordinate form of (28.1) is $[a:b:c] \mapsto [ab:ac:bc]$, so that (28.1) is a *quadratic transform with the centers* $[0:0:1]$, $[0:1:0]$ and $[1:0:0]$. (See [12, pp. 496–498].) In other words, (28.1) consists of a blow-up at the three centers, followed by a blow-down of the three projective lines through each pair of centers. The three centers lie in the projective closure $\overline{\mathcal{T}}$ of the curve $\mathcal{T}$ with the equation $T=0$, and one can show that $m-2$ is their common algebraic multiplicity. Hence they are singularities of $\overline{\mathcal{T}}$ for odd $m>3$, while $\overline{\mathcal{T}}$ also has a fourth singularity, of multiplicity $2(m-2)$, at the point $[a:b:c]$ with $a=b=-1$, $c=1$, that is, at the origin of the uv-plane.

Proposition 28.2 thus states that $\overline{\mathcal{T}}$ is invariant under a quadratic transform with centers which, for odd $m>3$, are its low-order singularities.

Remark 28.4. A well-known involution (cf. [1]) assigns to a quadruple (M,g,m,τ) with (0.1) or (0.2), such that g is not locally reducible as a Kähler metric, its *dual* $(\hat M, \hat g, \hat m, \hat\tau)$. Here $\hat M, M$ coincide as real manifolds, but have different complex structures, so that $\hat m = m$, while $\hat g = c^2 g/(\tau - c)^2$ and $\hat\tau = c\tau/(\tau - c)$, for c determined by (M,g,m,τ) as in Remark 3.2. (Cf. Theorem 1.3 and §43.) This duality involution induces the mapping (28.1) on $\mathcal{C} \smallsetminus \{(0,0)\}$ when (M,g,m,τ) is replaced by the point $(u,v) \in \mathcal{C}$ associated with it as in §0 or Remark 3.2.

29. Equation $T(u,v) = 0$ with $u < 0 < v < 1$

Proposition 29.1. *Given an integer $m \geqslant 2$, let Γ be the set of all $(u,v) \in \mathbf{R}^2$ with $u < 0 < v < 1$ and $T(u,v) = 0$, for T as in Lemma 17.3.*

- i) *If m is even, Γ is empty.*
- ii) *If m is odd, Γ is a real-analytic submanifold of $\mathbf{R}^2$, diffeomorphic to $\mathbf{R}$, containing the point (x,x^*) described in Lemma 25.2(b), and invariant under the involution (28.1). Furthermore, (28.1) keeps (x,x^*) fixed and interchanges the two connected components of $\Gamma \smallsetminus \{(x,x^*)\}$. One of these components is unbounded, while the closure of the other is the compact curve*

segment Z appearing in Theorem 27.3(b). *For q and q_* as in* (27.1), *the function q sends either component homeomorphically onto $(-1,q_*)$, while its value at (x,x^*) is q_*.*

Proof. Let $K=Z\cap(\mathbf{R}\times\{1\})$, with Z as in Theorem 27.3. Thus, $\varGamma$ is the union of the set $Z\smallsetminus K$ and its image under the involution (28.1). (In fact, Proposition 28.2(c) and Lemma 20.1 show that (28.1) sends the set $(-\infty,0)\times(0,1)$ onto itself by interchanging its subsets lying "above" and "below" the hyperbola $v=u^*$ and keeping each point of the hyperbola fixed.) Now (i) follows from Theorem 27.3(a).

If m is odd, Theorem 27.3 gives $K=\{(y,1)\}$, while both $Z\smallsetminus\{(y,1)\}$ and its image under (28.1) contain (x,x^*) (Theorem 27.3(b)), so that their union $\varGamma$ is connected. Thus, according to Remark 26.4, $\varGamma$ is contained in $\mathbf{R}^2$ as a (connected) one-dimensional real-analytic submanifold without boundary. Finally, by (20.2.a), $v^*\to-\infty$ and $u^*\to y^*$ as $(u,v)\in Z\smallsetminus\{(y,1)\}$ approaches $(y,1)$, which shows that $\varGamma$ is unbounded. This completes the proof. ☐

Lemma 29.2. *Given an integer $m\geqslant 3$, let T,λ,A,B be the rational functions of u,v defined in* Lemma 17.3, (5.8) *and* (10.3) *with $E,F,\varSigma$ as in* (2.1) – (2.2), *and let μ be the rational function of u,v with $\mu=uv/(u+v-2)$. Then*

$$\mu-\lambda=\frac{(v-u)^3u(u-2)v(v-2)T(u,v)}{m(u+v-2)(u-1)^m(v-1)^mA}\tag{29.1}$$

in the sense of equality between rational functions. Also, $0<\mu<1$ at any $(u,v)\in(-\infty,0)\times(0,1]$. Finally, when m is odd and $(u,v)\in(-\infty,0)\times(0,1)$,

i) *$A<0$ at (u,v), with A as in* (10.3).
ii) *$T(u,v)=0$ if and only if $\lambda=\mu$ at (u,v).*

In fact, $0<\mu<1$ on $(-\infty,0)\times(0,1]$ as $(u-1)(v-1)\geqslant 0>-1$ there, and hence also $u+v-2<uv<0$. Next, dividing (24.3.i) by $m\alpha\phi$, with α,ϕ as in (24.1), and using (24.2), (24.3.ii) and (5.8), we obtain (29.1). Finally, if m is odd, $F(u)<0<F(v)$ as $u<0<v<1$ (see (6.1)). Thus, $e_u-e_v>0$ for $e_u=E(u)/F(u)$, $e_v=E(v)/F(v)$ (as $e_v<0<1<e_u$ by (6.1)), and so $A=(e_u-e_v)F(u)F(v)<0$, which proves (i). Now (ii) follows: the denominators involved (including those in (5.8)), and the factor $(v-u)u(u-2)v(v-2)$, are all nonzero when $u<0<v<1$. (Cf. (i) and the obvious relation $u+v-2<0$.)

Proposition 29.3. *Given an odd integer $m\geqslant 3$ and real numbers u,v with $u<0<v<1$, let T be the polynomial defined by* (17.5), *and let $Q\in\mathbf{V}\smallsetminus\{0\}$ be a function with* (2.5.b) *on the interval $I=[u,v]$, where $\mathbf{V}$ is the space* (2.4). *In view of* Lemma 9.1, *such Q exists and is unique up to a nonzero constant factor.*

If $T(u,v)=0$, then Q and I satisfy all five conditions (2.5).

In fact, $\lambda=\mu>0$ at (u,v) in view of Lemma 29.2(ii) and relation $\mu>0$ in Lemma 29.2. Thus, by Lemma 29.2(i) and (5.8), $\lambda A<0$ and $B\varSigma(0)>A$ at (u,v), with $A,B,\varSigma$ given by (10.3), (2.2). However, (10.3) also yields $B<0$, as F is strictly increasing on $(-\infty,1)$ (see §6). Hence $A/B>\varSigma(0)=-E(0)$, cf. (5.6.ii). Proposition 15.2 now implies that Q and I satisfy (2.5.a), (2.5.c) and (2.5.d), while relation $T(u,v)=0$ yields (2.5.e) (see Theorem 17.5).

30. Expansions of T about (0,0) and (1,1)

For E,c as in (2.1) and (24.3.ii) with an integer $m\geqslant 2$, we have $c=\binom{2m-1}{m-1}$ (see (5.6.i,iii)), and there exists a polynomial D such that, for all $u\in\mathbf{R}$,

$$\begin{aligned}&\text{a)}\quad (u-1)^mE(u)=(u-2)u^{2m-1}+D(u),\\&\text{b)}\quad D(u)-c(u-1)^m=(u-2)\textstyle\sum_{j=0}^{m-1}(-1)^{m+j}\binom{2m-1}{j}\binom{2m-j-2}{m-1}u^j.\end{aligned}\tag{30.1}$$

Namely, $D_1(u) = 1$ and, by (5.2.ii), $D_m(u) = u^2 D_{m-1}(u) - m^{-1}\binom{2m-2}{m-1}(u-1)^m$ whenever $m \geqslant 2$, for $D = D_m$ *defined* by (30.1.a). Now $D(u) = m^{-1}\sum_{j=0}^{m-1}(-1)^{m+j+1}\binom{2m}{j}\binom{2m-j-2}{m-1}u^j$, which follows from easy induction on m, and amounts to (30.1.b).

Lemma 30.1. *Every symmetric polynomial in the variables u, v can be uniquely written as a combination of the products $(uv)^j\,\Theta_k(u,v)$, where $j \geqslant 0$ and $k \geqslant 2$ are integers, and $\Theta_k(u,v) = \sum_{j=1}^{k-1} j\,(k-j)\,u^{j-1}v^{k-j-1}$.*

In fact, the space of degree m homogeneous symmetric polynomials in u, v has the obvious basis $\Phi_\rho = (uv)^\rho(u^{m-2\rho} + v^{m-2\rho})$ with $\rho \in \mathbf{Z}$ and $0 \leqslant \rho \leqslant m/2$. Then $\tilde\Phi_\rho = (uv)^\rho\,\Theta_{m-2\rho+2}(u,v)$ form another basis of that space. Namely, $\tilde\Phi_\rho$ equals $(m-\rho+1)\Phi_\rho$ plus a combination of Φ_σ with $\rho < \sigma \leqslant m/2$, as one sees pairing up, for each j, the jth and $(k-j)$th terms in the formula for $\Theta_k(u,v)$. Thus, the triangular matrix expressing the $\tilde\Phi_\rho$ through the Φ_ρ is invertible.

Remark 30.2. By Lemma 30.1, any symmetric polynomial Φ in the variables a, b has an expansion $\sum_{j,k}\mathbf{c}_{j,k}(ab)^j\,\Theta_k(a,b)$ with some unique coefficients $\mathbf{c}_{j,k}$, indexed by integers j, k, and such that $\mathbf{c}_{j,k} \neq 0$ for at most finitely many pairs j, k, all of which have $j \geqslant 0$ and $k \geqslant 2$. Expanding $(a-b)^3\,\Theta_k(a,b)$ into powers of a and b, we get

$$(a-b)^3\,\Theta_k(a,b) \,=\, (k-1)a^{k+1} - (k+1)a^k b + (k+1)ab^k - (k-1)b^{k+1}, \tag{30.2}$$

and so, for any $\rho, \sigma \in \mathbf{Z}$, the coefficient of $a^\rho b^\sigma$ in the monomial expansion of $(b-a)^3\Phi$ is

$$\begin{aligned}(\sigma-\rho-2)\mathbf{c}_{\rho,\sigma-\rho-1} \;&+\; (\rho-\sigma-2)\mathbf{c}_{\rho-1,\sigma-\rho+1}\\ &+\; (\sigma-\rho+2)\mathbf{c}_{\sigma,\rho-\sigma-1} \;+\; (\rho-\sigma+2)\mathbf{c}_{\sigma-1,\rho-\sigma+1}\,.\end{aligned} \tag{30.3}$$

Lemma 30.3. *Let $L(u,v) = E(u)f - c\phi$ with E, c as in* (2.1), (24.3.ii) *and ϕ, f depending on $u, v \in \mathbf{R}$ as in* (24.1) *for a fixed integer $m \geqslant 2$. Then, for $a, b \in \mathbf{R}$,*

i) $E(a+1) \,=\, 2\sum_{j=1}^m j(m+j)^{-1}\gamma(j)a^j$, *and*

ii) $L(a+1, b+1) = (a^2-1)\sum_{j=1}^m (jb-j+1)\gamma(j)a^{j-1}$, *where* $\gamma(j) = \binom{2m-1}{m-j}$.

Proof. We prove (i) by induction on $m \geqslant 1$, writing, as in (5.2), E_m, γ_m for E, γ. If $m = 1$, (i) is trivial as $E_1(a+1) = a$. The inductive step: (5.2.ii) multiplied by $2m-1$ gives $(2m-1)E_m(a+1) = (2m-1)a^{-1}(a+1)^2E_{m-1}(a+1) - \gamma_m(1)$. Assuming (i) for $m-1$ rather than the given $m \geqslant 2$, let us first substitute for $E_{m-1}(a+1)$, in the last equality, the sum as in (i) involving the $\gamma_{m-1}(j)$ instead of $\gamma_m(j)$, then multiply by $(a+1)^2$ and make the summation index j coincide with the exponent in a^j, next replace $(2m-1)(m+j-1)^{-1}\gamma_{m-1}(j)$ by $(2m-2)^{-1}(m-j)\gamma_m(j)$, as well as $(m-j+1)\gamma_m(j-1)$ by $(m+j-1)\gamma_m(j)$ and $\gamma_m(j+1)$ by $(m+j)^{-1}(m-j)\gamma_m(j)$. This yields (i) for m, as required. Assertion (ii) will in turn follow once we establish the identities

iii) $(m+a-ma)E(a+1) \,-\, a\gamma(1) \,=\, (a^2-1)\sum_{j=1}^m (1-j)\gamma(j)a^{j-1}$,

iv) $(ma-m+1)E(a+1) \,-\, \gamma(1) \,=\, (a^2-1)\sum_{j=1}^m j\gamma(j)a^{j-1}$.

In fact, to get (ii) one can add (iv) multiplied by b to (iii), as (24.1.c) for $(u,v) = (a+1, b+1)$ gives $f = (ma-m+1)b + (m+a-ma)$ and $\phi = a+b$.

To prove (iii) and (iv), we use (i) to rewrite all four expressions as combinations of the powers a^j and verify that the corresponding coefficients agree. Such a coefficient, found by expressing $\gamma(j\pm1)$ through $\gamma(j)$ (as above, with the subscript m), turns out to be $0, \gamma(2), 1-m$ for $j = 0, 1, m+1$ in (iii) and $-\gamma(1), m$ for $j = 0, m+1$ in (iv), while for (iii) and $j = 2, \ldots, m$ (or, respectively, (iv) and $j = 1, \ldots, m$) it equals $2\gamma(j)/[(m+j)(m-j+1)]$ times $3mj - 2mj^2 + m^2 + j^2 - m - j$ (or, respectively, times $2mj^2 - mj - m^2 - j^2 + j$). This completes the proof. ∎

Lemma 30.4. *Let $m \geqslant 2$ be an integer. For the polynomials T, Θ_k defined as in* Lemmas 17.3 *and* 30.1, *and $u, v, a, b \in \mathbf{R}$, we then have* $\deg T = 3(m-2)$ *and*

a) $T(u,v) \;=\; \dfrac{1}{m-1}\displaystyle\sum_{k=m}^{2m-2}(-1)^{m+k}\binom{2m-1}{k+1}\binom{k-2}{m-2}(uv)^{2m-k-2}\Theta_k(u,v).$

b) $T(a+1,b+1) \;=\; \displaystyle\sum_{j=0}^{m-2}\sum_{k=m-j}^{2m-j-2}\mathbf{c}_{jk}(ab)^j\Theta_k(a,b)$, *with* $\mathbf{c}_{jk}$ *given by*

c) $\mathbf{c}_{jk} \;=\; \dfrac{(2m-1)(m-j-1)(k-m+j+1)}{(m-1)(k^2-1)}\dbinom{2m-2}{j}\dbinom{2m-2}{k+j}.$

Proof. Formula (24.3.i) with $(u,v)=(a+1,b+1)$ gives $(b-a)^3(a^2-1)(b^2-1)T(a+1,b+1)=(b^2-1)(b+1)^{2m-2}a^mL(a+1,b+1)-(a^2-1)(a+1)^{2m-2}b^mL(b+1,a+1)$, for L as in Lemma 30.3. By Lemma 30.3(ii), this polynomial equality yields

$$\begin{array}{rl} \text{i)} & (b-a)^3T(a+1,b+1) \;=\; \Psi(a,b)\;-\;\Psi(b,a), \quad \text{as well as}\\ \text{ii)} & \Psi(a,b)=\sum_{\rho=m}^{2m-1}\sum_{\sigma=0}^{2m-1}\mathbf{d}_{\rho\sigma}a^\rho b^\sigma, \quad \text{where } \Psi \text{ is the polynomial with}\\ \text{iii)} & (a^2-1)\Psi(a,b) \;=\; (b+1)^{2m-2}a^mL(a+1,b+1), \quad \text{and}\\ \text{iv)} & \mathbf{d}_{\rho\sigma} \;=\; \binom{2m-1}{\rho}\binom{2m-1}{\sigma}[m-(\rho+\sigma)+2\rho\sigma/(2m-1)]. \end{array} \tag{30.4}$$

According to Lemma 30.1, $T(a+1,b+1)=\sum_{j,k}\mathbf{c}_{j,k}(ab)^j\Theta_k(a,b)$ with $\mathbf{c}_{j,k}$ as in Remark 30.2. In terms of the conditions

$$\text{i)}\quad 0\leqslant\sigma\leqslant m-1<\rho<2m, \qquad \text{ii)}\quad 0\leqslant\rho\leqslant m-1<\sigma<2m \tag{30.5}$$

for $\rho,\sigma\in\mathbf{Z}$, these $\mathbf{c}_{j,k}$ satisfy the following system of linear equations:

$$\begin{array}{l}\text{expression (30.3) equals:}\\ \mathbf{d}_{\rho\sigma} \text{ in case (30.5.i)}, \quad -\,\mathbf{d}_{\rho\sigma} \text{ in case (30.5.ii)}, \text{ and } 0 \text{ otherwise,}\end{array} \tag{30.6}$$

with $\rho,\sigma\in\mathbf{Z}$ and $\mathbf{d}_{\rho\sigma}$ as in (30.4.iv); see (30.4.i) and Remark 30.2. Moreover, the $\mathbf{c}_{j,k}$ form the *unique* solution to (30.6), under the finiteness requirement of Remark 30.2. (The coefficients of $a^\rho b^\sigma$ in the expansion of the right-hand side (30.4.i) are zero for those ρ,σ which both lie in the range $m,\dots,2m-1$, due to the subtraction in (30.4.i) and the obvious symmetry relation $\mathbf{d}_{\rho\sigma}=\mathbf{d}_{\sigma\rho}$.)

We now show that by setting $\mathbf{c}_{j,k}=\mathbf{c}_{jk}$ for $\mathbf{c}_{jk}$ as in (c) when j and $j+k-m$ lie in $\{0,\dots,m-2\}$, and $\mathbf{c}_{j,k}=0$ otherwise, one obtains a solution to (30.6).

First, let us assume (30.5.i). If $\sigma=0$, all but the third term on the left-hand side then must vanish, as they involve $\mathbf{c}_{j,k}$ with $k<0$ or $j<0$, and we get (30.6) by considering the separate cases $m=\rho=2$ and $\rho>2$. If $\sigma=m-1$, only the fourth term may be nonzero (as the others involve $\mathbf{c}_{j,k}$ with $k<0$ or $k=0$ or $j=m-1$), and (30.6) is easily verified. If, however, $0<\sigma<m-1<\rho<2m$, the first two terms vanish as they have $j>m-2$, and two cases are possible: $\rho=m=\sigma+2$ (so that the third term, with $k=1$, is zero, and (30.6) easily follows), or $\rho-\sigma>2$, and a simple calculation again gives (30.6).

Now let (30.5.ii) be satisfied. Interchanging ρ and σ we reduce this case to (30.5.i), since both sides of (30.6) are antisymmetric in ρ,σ (as the second line arises from the first by switching ρ,σ and changing the sign, while $\mathbf{d}_{\rho\sigma}=\mathbf{d}_{\sigma\rho}$).

However, if we have neither (30.5.i) nor (30.5.ii), the right-hand side vanishes, and so do all four terms on the left-hand side, as our bounds on j and $j+k-m$ in the definition of $\mathbf{c}_{j,k}$ show that condition (30.5.i) (or, (30.5.ii)) is necessary for the second (or, first) line to contain nonzero term. Uniqueness of the solution to (30.6) now proves (b).

Next, with c,ϕ,f and $L(u,v)=E(u)f-c\phi$ as in Lemma 30.3, (30.4.iii) for $(a,b)=(u-1,v-1)$ and (30.1.a) show that $(u-2)u\Psi(u-1,v-1)$ equals $v^{2m-2}[(u-2)u^{2m-1}+D(u)-c(u-1)^m]f+cv^{2m-2}(u-1)^m(f-\phi)$. Using (30.1.b) and the relation $f-\phi=m(u-2)(v-2)$, we can now divide both sides by $u-2$, obtaining a polynomial expression for $u\Psi(u-1,v-1)$ that involves

two expansions into powers of u, coming from (30.1.b) and the binomial formula for $(u-1)^m$. As $mc=(2m-1)\binom{2m-2}{m-1}$ by (24.3.ii) and (5.6.i), the overall coefficient of the 0th power of u is zero and so we can divide both sides by u, getting

$$\begin{aligned}\varPsi(u-1,v-1) &= (uv)^{2m-2}[muv-(2m-1)(u+v-2)]\\ &+ \textstyle\sum_{j=0}^{m-2}(-1)^{m+j}\binom{2m-1}{j}\binom{2m-j-3}{m-1}u^jv^{2m-1}\\ &+ \tfrac{2m-1}{m-1}\textstyle\sum_{j=1}^{m-1}(-1)^{m+j}\binom{2m-2}{j-1}\binom{2m-j-3}{m-2}u^jv^{2m-2}.\end{aligned}\tag{30.7}$$

Assertion (a) claims that $T=\varPhi$ for a polynomial $\varPhi$ given by $\varPhi(u,v)=\sum_{j,k}\mathbf{c}_{j,k}(uv)^j\varTheta_k(u,v)$ with a specific *new meaning* of the coefficients $\mathbf{c}_{j,k}$. Clearly, (a) will follow if we prove that the equality $(v-u)^3T(u,v)=\varPsi(u-1,v-1)-\varPsi(v-1,u-1)$, immediate from (30.4.i), still holds when T is replaced by $\varPhi$. This, according to Remark 30.2, amounts to showing that (30.3), for any $\rho,\sigma\in\mathbf{Z}$, equals the coefficient of $u^\rho v^\sigma$ in the expansion of $\varPsi(u-1,v-1)-\varPsi(v-1,u-1)$. The latter coefficient is easily obtained from the right-hand side of (30.7), with no contribution from the first line (which is symmetric in u,v). Also, $\mathbf{c}_{j,k}$ are nonzero only for integers j,k with $j+k=2m-2$ and $0\leqslant j\leqslant m-2$, and so, for any given $\rho,\sigma\in\mathbf{Z}$, at most one of the four expressions $\mathbf{c}_{j,k}$ occurring in (30.3) may be nonzero. Specifically, the first (or second, third, fourth) $\mathbf{c}_{j,k}$ in (30.3) is nonzero only if $0\leqslant\rho\leqslant m-2$ and $\sigma=2m-1$, or $1\leqslant\rho\leqslant m-1$ and $\sigma=2m-2$, or $0\leqslant\sigma\leqslant m-2$ and $\rho=2m-1$, or, respectively, $1\leqslant\sigma\leqslant m-1$ and $\rho=2m-2$. The required equality is easily verified in each of these four cases, while both sides are zero when none of the four cases occurs. We have thus established (a). Finally, $\deg T=3(m-2)$ as the kth term in (a) is homogeneous of degree $4m-k-6$. This completes the proof. ∎

Remark 30.5. Let $m=3$. Lemma 30.4(a) then yields $T(u,v)=5(uv^2+vu^2)-(3u^2+3v^2+4uv)=2[5(\xi-1)\xi^2-(5\xi+1)\eta^2]$, with ξ,η given by (17.1). Fig. 1 thus conveys the correct idea of what the set given by $T(u,v)=0$ looks like when $m=3$. Namely, the lines $v=u$ and $v=-u$ are the ξ and η coordinate axes, while $T(u,v)=0$ amounts to $\xi\neq-1/5$ and $\eta^2=(\xi-1)\xi^2/(\xi+1/5))$. This gives $\xi\geqslant1$ or $\xi<-1/5$, and $\eta=\pm\xi[1-6/(5\xi+1)]^{1/2}$.

31. The sign of T on specific regions

Assertion (i) below is needed only to derive (ii) and (iii); however, (ii), (iii) also follow from our formula (30.2) combined with Lemma 32.1 of [11] for $\beta=u/v$.

Lemma 31.1. *For an integer $k\geqslant2$, any $u,v\in\mathbf{R}$, and $\varTheta_k$ as in* Lemma 30.1,

i) $\varTheta_k(u,v)=\sum_{1\leqslant j\leqslant k/2}j\binom{k+1}{2j+1}\xi^{k-2j}\eta^{2j-2}$ *for ξ,η given by* (17.1).

ii) *If k is even, $\varTheta_k(u,v)>0$ unless $u=v=0$, while $\varTheta_k(0,0)=0$.*

iii) *If k is odd, $\operatorname{sgn}(\varTheta_k(u,v))=\operatorname{sgn}(u+v)$, with sgn as in* §4.

Here (i) is straightforward if one multiplies both sides by $(u-v)^3=8\eta^3$ and verifies that the resulting polynomials in ξ,η coincide by applying (17.1) and the binomial formula to rewrite the right-hand side of (30.2) as a function of ξ,η. Now (ii) and (iii) follow since, when k is even (or, odd), (i) expresses $\varTheta_k(u,v)$ as a sum of squares (or, respectively, as ξ times a sum of squares).

Let T be the polynomial defined by (17.5) with any given integer $m\geqslant2$. Then $T=1$ for $m=2$ (cf. Lemma 30.4(a)), while, if $m\geqslant3$,

$$\begin{array}{llll}\text{a)} & \operatorname{sgn}T=(-1)^m & \text{at any } (u,v)\neq(0,0) \text{ with } (u+v)uv\leqslant0,\\ \text{b)} & \operatorname{sgn}T=(-1)^m & \text{on } [0,1)\times[0,1) \text{ except at } (0,0) \text{ or } (1,1),\\ \text{c)} & \operatorname{sgn}T=(-1)^m & \text{on } (-\infty,0]\times(-\infty,0] \text{ except at } (0,0),\\ \text{d)} & \operatorname{sgn}T=1 & \text{on } [1,\infty)\times[1,\infty) \text{ except at } (1,1).\end{array}\tag{31.1}$$

In fact, let $u+v \geqslant 0$ and $uv \leqslant 0$, or $u+v \leqslant 0$ and $uv \geqslant 0$. The even k (or, odd k) summands in Lemma 30.4(a) are nonnegative by Lemma 31.1(ii) (or, respectively, Lemma 31.1(iii)). This yields (31.1.a), and hence (31.1.c). Next, as $u^*, v^* \in (-\infty, 0]$ whenever $u, v \in [0,1)$ (cf. Lemma 20.1), (31.1.c) and Proposition 28.2(b) imply (31.1.b). Finally, Lemmas 30.4(b) and 31.1(ii), (iii) give (31.1.d).

32. Third subcase of (2.5.e): condition (17.6.c)

Let $\mathbf{V}$ be the space (2.4) for a given integer $m \geqslant 2$, and let us fix $u, v \in \mathbf{R}$ with $u \neq v$. By Lemma 9.1, there exists $Q \in \mathbf{V} \smallsetminus \{0\}$ satisfying (2.5.b) on the interval I with the endpoints u, v, and such Q is unique up to a constant factor.

For these Q, I, (2.5.e) holds if and only if (u,v) satisfies one of the three conditions in (17.6). (See Theorem 17.5.) The question of finding direct descriptions of the three sets in the uv-plane $\mathbf{R}^2$ defined by (17.6.a), (17.6.b) and, respectively, (17.6.c), has an obvious answer for (17.6.b), the set being the hyperbola $v = u^*$ (that is, $uv = u + v$). As for (17.6.a), Proposition 18.2 yields an answer: (17.6.a) defines for odd m the two-point set $\{(1,s),(s,1)\}$, and for even m the empty set.

Our real interest lies, however, in those (u,v) for which Q, I chosen above satisfy *all five* conditions in (2.5). This leads to switching our focus from the three solution sets in $\mathbf{R}^2$ (see the last paragraph) to their respective subsets obtained by imposing on Q, I also the positivity condition of §16. For (17.6.a), the resulting subset is empty (Proposition 19.2), while in the case of (17.6.b) an explicit description of that subset is provided by Proposition 23.1.

That new focus also explains why, unlike the approach to (17.6.a) – (17.6.b) outlined above, our discussion of (17.6.c) bypasses the step of first describing the set given by (17.6.c) alone. Instead, we proceed directly to discuss the subset of the uv-plane $\mathbf{R}^2$ defined by requiring that Q, I corresponding to the given (u,v) with $u \neq v$ satisfy *both* (17.6.c) *and* (2.5). This subset turns out to be empty for even m, while for odd m it is the union of the curve $\Gamma \subset (-\infty, 0) \times (0,1)$ described in Proposition 29.1(ii) and the image of Γ under the symmetry $(u,v) \mapsto (v,u)$.

In fact, a point (u,v) in this subset can never satisfy any of the following conditions: $u = 1$ or $v = 1$ (by (17.6.c)), $uv = 0$, or $u < 1 < v$, or $v < 1 < u$ (see (ii) in §16). Also, u, v cannot both lie in $(-\infty, 0)$, $(0,1)$, or $(1,\infty)$ (by (17.6.c) and (31.1.b-d)). This leaves $u < 0 < v < 1$ or $v < 0 < u < 1$ as the only possibilities, so that our claim follows from Proposition 29.1 and symmetry of T.

33. A synopsis of conditions (2.5)

The *moduli curve* defined in §2 is the subset $\mathcal{C}$ of $\mathbf{R}^2$, depending on an integer $m \geqslant 2$, and formed by all pairs (u,v) such that either $(u,v) = (0,0)$, or $u < v$ and all five conditions in (2.5) are satisfied by the interval $I = [u,v]$ and some function $Q \in \mathbf{V} \smallsetminus \{0\}$, with $\mathbf{V}$ as in (2.4). In view of Lemma 9.1, Q then is unique up to a constant factor, while Lemma 10.1 and Remark 12.2 provide a choice of such Q for which A, B, C in (2.3) are specific rational functions of u, v. Namely, they are given by (10.3) (if $1 \notin I$), or by $(A,B,C) = (-E(t), 1, 0)$ (if $\{u,v\} = \{1,t\}$).

The next result clearly implies (1.1) for the sets I, $\mathsf{!}$ and X defined in §1. In other words, the definition of $\mathcal{C}$ (see above) agrees with the explicit description $\mathcal{C}$ given in §1.

THEOREM 33.1. *Let $u^* = u/(u-1)$ for $u \neq 1$. A pair $(u,v) \in \mathbf{R}^2$ lies in the moduli curve $\mathcal{C}$, defined as above for a fixed integer $m \geqslant 2$, if and only if one of the following three cases occurs:*

a) *m is even, $u \in (-\infty, 0] \cup (1,2)$, and $v = u^*$.*

b) *m is odd, $u \in (-\infty, z) \cup (w, 0] \cup (1,2)$, and $v = u^*$, for z, w as in §21.*

c) *m is odd and $(u,v) \in \varGamma$, where $\varGamma \subset (-\infty,0) \times (0,1)$ is the connected real-analytic curve diffeomorphic to* $\mathbf{R}$, *described in* Proposition 29.1(ii).

Proof. Let $u < v$. Each of (a), (b), (c) separately implies (2.5) with Q, I as above. For (a), (b) this is clear: $1 \notin I$ (cf. (20.2)), which yields (2.5.a); $v = u^*$, i.e., (17.6.b), implies (2.5.e) (Theorem 17.5); Proposition 23.1 and Lemma 20.1 give (2.5.c) – (2.5.d). Also, (c) yields (2.5) by Proposition 29.3.

Conversely, let Q, I as above satisfy (2.5). Theorem 17.5 then gives (17.6.b) or (17.6.c), as Proposition 19.2 excludes (17.6.a). In case (17.6.b), we must have (a) or (b) (Proposition 23.1 and Lemma 20.1), while (17.6.c) yields (c), cf. the fourth paragraph of §32. This completes the proof. $\square$

Corollary 33.2. *If the integer $m \geqslant 2$ is even, or odd, then the two sets $\mathsf{!}$, I or, respectively, three sets X, $\mathsf{!}$, I, defined in* §1, *are the connected components of the moduli curve $\mathcal{C}$.*

This is clear from (1.1): the two/three sets are connected, relatively open in $\mathcal{C}$ and, for odd m, the $\mathcal{T}$-beam $\mathsf{X} \cap \mathcal{T}$ coincides with the set $\varGamma$ in Proposition 29.1, and so, by Lemma 25.2(b), $\mathsf{X} \cap \mathcal{T}$ intersects the $\mathcal{H}$-beam $\mathsf{X} \cap \mathcal{H}$ only at (x, x^*), and does not intersect I or $\mathsf{!}$.

Remark 33.3. For odd m, the intersection of the two beams of X at their unique common point (x, x^*) is transverse, by Lemma 25.2(d).

Remark 33.4. Let (2.5) be satisfied by $I = [u, v]$ and a rational function Q of the form (2.3) with some $A, B, C \in \mathbf{R}$. Then $C \neq 0$, so that, by (2.1), Q is analytic on $\mathbf{R} \smallsetminus \{1\}$ and has a pole at 1.

In fact, suppose on the contrary that $C = 0$. Thus, m is odd and $(u,v) \in \varGamma$, for $\varGamma$ as in Proposition 29.1(ii) (or else, as our assumption gives $(u,v) \in \mathcal{C} \smallsetminus \{(0,0)\}$, Theorem 33.1 would yield $v = u^*$ with $0 \neq u \neq 2$, and hence $C \neq 0$ by (21.3) and (2.1)). Also, (2.5.b), (2.5.c) and (2.3) with $C = 0$ imply (12.1) with $B \neq 0$. Therefore, since $u \neq 1 \neq v$, Lemma 12.1(i) shows that $E(u) = E(v) = -A/B$ as well as $\dot{Q}(t) = (t-1)B\dot{E}(t)$, with $(\ )^{\boldsymbol{\cdot}} = d/dt$, for both $t = u$ and $t = v$. Now (2.5.e) yields $(u-1)\dot{E}(u) = (1-v)\dot{E}(v)$. Since $u < 0 < v < 1$ (due to the definition of $\varGamma$), (13.2) along with the three lines preceding it (from now on referred to simply as §13) give $E(0) < E(v) < 0 < \dot{E}(v)$, while $u - 1 < 0 < 1 - v$. Thus, by the last equality, $\dot{E}(u) < 0$, and $E(0) < E(u) < 0$ as $E(u) = E(v)$. From $u < 0$ and $\dot{E}(u) < 0$ we in turn get $u > \tilde{w} > \tilde{z}$ (see §13), and so $u > \tilde{w} > x$, as $\tilde{z} > z$ (Remark 21.1) and $z > x$ (Lemma 25.2(b)). Similarly, using (20.2.a) for $t = \tilde{w}$ and noting that $x^* > t^*$ as $t = \tilde{w} > x$, cf. Lemma 20.1, while $F < 0$ on $(-\infty, 0)$ (see §6), we get $E(x^*) > E(t^*) = E(t) - F(t) > E(t) > E(u) = E(v)$ for $t = \tilde{w}$ from (20.1.ii) and the monotonicity properties of E listed in §13, and so, again from §13 and Lemma 20.1, $0 < v < x^* < 1$.

The function q given by (27.1) is clearly increasing (or, decreasing) as a function of u (or, v) alone in the region where $u < 0 < v < 1$. As $x < u < 0$ and $0 < v < x^*$, this implies that the value of q at (u,v) is *greater* than q_*, its value at (x, x^*). Since $(u,v) \in \varGamma$ and $q \leqslant q_*$ on $\varGamma$ (see the last sentence in Proposition 29.1(ii)), we now obtain a contradiction. Therefore, $C \neq 0$.

34. The rational function p

Given an integer $m \geqslant 2$, we let p stand for the rational function of the variables u, v, defined by the formula in (5.7) with λ as in (5.8) for $A, B, \varSigma$ given by (10.3) and (2.1) – (2.2). For later convenience, we modify this definition by declaring the value of p at $(0,0)$ to be 0.

Thus, p is real-analytic everywhere in $\mathbf{R}^2$ with a possible exception of those points $(u,v) \in \mathbf{R}^2$ for which $u \in \{0,2\}$ (cf. (5.7), or $u = 1$, or $v = 1$ (since (5.8) involves A, B with (10.3), and F, given by (2.1), has a pole at 1), or, finally, $u \neq 1 \neq v$ and $A = 0$ at (u,v) (as A appears in the denominator of (5.8)).

We are interested in the restriction of p to the moduli curve $\mathcal{C}$ (see §33). Of the singularities just listed, only $(0,0)$ lies on $\mathcal{C}$ if m is odd, and just two, $(0,0)$ and (z, z^*), lie on $\mathcal{C}$ when m

is even, with $z < 0$ defined as in §21 and $z^* = z/(z-1)$. The singularity at (z, z^*) arises since $A = 0$ there.

In fact, for $(u, v) \in \mathcal{C} \smallsetminus \{(0,0)\}$ we have $u, v \notin \{0, 1, 2\}$ by Theorem 33.1 and Lemma 20.1. Hence, from (10.3) and (2.1), $A = 0$ at such (u, v) if and only if E/F has equal values at u and v. This in turn excludes the possibility that either $1 < u < 2 < v$, or m is odd and $u < 0 < v < 1$ (cf. the last two lines in (6.1)), so that, by Lemma 20.1, the only case still allowed in Theorem 33.1 is (a) with $u \in (-\infty, 0]$ (and so m is even, while $v = u^*$). Our claim about (z, z^*) and A now follows from (21.3) – (21.5) (and (2.1)).

Remark 34.1. Given an integer $m \geqslant 2$, we have $\kappa = \varepsilon m A/c$ and (4.1) whenever $(u, v) \in \mathcal{C} \smallsetminus \{(0,0)\}$ and Q is a function in the space (2.4) satisfying (2.5) on $I = [u, v]$, while $A \in \mathbf{R}$ is determined by Q via (2.3), $\varepsilon = \pm 1$ and c, a are nonzero constants with $dQ/dt = -2ac$ at $t = u$, and, finally, either $a = -\varepsilon p\kappa/2$ (where p stands for the value at (u, v) of the rational function p), or m is even, $(u, v) = (z, z^*)$ and $\kappa = 0$. In fact, if m is even and $(u, v) = (z, z^*)$, this follows since, as we just saw, A then equals 0 at (z, z^*). Otherwise, $A \neq 0$ at (u, v) and $u, v \notin \{0, 1, 2\}$ (see above), so that, by (4.1.i) and (2.5.d), we have $p \neq 0$ and $mp = \dot{Q}(u)/A = -2ac/A = \varepsilon p\kappa c/A$, as required.

By (1.1), a substantial part of the moduli curve $\mathcal{C}$ is contained in the hyperbola $\mathcal{H}$ given by $v = u^*$, where $u^* = u/(u-1)$. It is therefore useful to introduce a rational function P of the real variable u which is the restriction of p to $\mathcal{H}$, that is, the result of substituting u^* for v in the rational expression for p in terms of u and v. Then, with Σ, G as in (2.2) and (21.1), we have

$$u(u-2)\,P(u) \;=\; 2(2\,-\,1/m)(u-1)\,[1\,+\,\Sigma(0)/G(u)]\;-\;u^2, \tag{34.1}$$

in the sense of equality between rational functions of u. In fact, for those u for which $P(u)$ defined by (34.1) makes sense, it coincides with the number p in (5.7) for $\lambda = (2 - 1/m)[1 + \Sigma(0)/G(u)]$, that is, for λ given by (5.8) with $v = u^*$, as one sees evaluating B/A from (21.3).

A trivial argument (see (c) in §35) shows that the rational function P defined by (34.1) is analytic at 0 and $P(0) = 0$. Thus, our convention about p at $(0,0)$ requires no further modification of P.

Remark 34.2. Let q and q_* be as in (27.1) for a fixed odd integer $m \geqslant 3$. The $\mathcal{T}$-beam of the moduli curve (cf. §1) is the set $\varGamma$ in Proposition 29.1, while $p = q$ at any point $(u, v) \in \varGamma$ since, by Lemma 29.2(ii), λ in (5.7) then may be replaced with μ. Thus (cf. the final clause in Proposition 29.1(ii)), the value of p at (x, x^*) is $q_* \in (-1, 0)$.

35. Some properties of P

Let $\dot{P} = dP/du$ for the rational function P given by (34.1) with a fixed integer $m \geqslant 2$. Then P satisfies the differential equation

$$u(u-1)\,\dot{P} \;=\; muP^2 \;+\; (m-1)(u-2)\,P \;-\; u\,, \tag{35.1}$$

where P, $\dot{P}$ stand for $P(u)$, $\dot{P}(u)$. Also, for z, w as in §21 and $u^* = u/(u-1)$,

a) If m is odd, P is analytic everywhere in $\mathbf{R}$ and $P(w) = P(w^*) = 0$.
b) If m is even, P has just two real poles, at z and z^*.
c) $P(0) = P(2) = 0$, $P(1) = 1$, $\dot{P}(0) = 1/(3-2m)$, $\dot{P}(1) = 2(1-m)/m$.
d) $P(u^*) = -P(u)$ for all $u \in \mathbf{R} \smallsetminus \{1\}$ at which P is analytic.

In fact, (d) is immediate if one replaces u in (34.1) by u^* and then uses (20.2) and (21.2.a). If m is odd, $\dot{G}(w) = 0$ by the definition of w, and so (21.2.c) with $t = w$ gives $\Sigma(0)/G(w) = \varLambda(w)/[2(2m-1)(w-1)] = -1 + mw^2/[2(2m-1)(w-1)]$, where we also used (5.1.b). Replacing $\Sigma(0)/G(u)$ in (34.1) by this last expression for $u = w$, we obtain $P(w) = 0$ (as $w < 0$) and, from

(d), $P(w^*) = 0$. Next, dividing (34.1) by $u(u-2)$ we find, using l'Hospital's rule and (21.2.d) – (21.2.f), that $P(u) \to 0$ as $u \to 0$ or $u \to 2$. Thus, $P(0) = P(2) = 0$. The comment on the zeros of G following (21.5) now gives $P(1) = 1$ (by (34.1)), and hence (a), (b).

With G expressed in terms of the function $\varphi(u) = 2(2m-1)(u-1)\Sigma(0)/G(u)$, equation (21.2.c) becomes $u(u-1)(u-2)\dot\varphi + [\Lambda(u) - \varphi - u(u-2)]\varphi = 0$, where $\varphi, \dot\varphi$ stand for $\varphi(u), \dot\varphi(u)$. Replacing φ by $\Lambda(u) + mu(u-2)P$ (which equals φ in view of (34.1) and (5.1.b)), and then substituting for $\Lambda(u)$ the expression in (5.1.b), we can further rewrite this as an equation imposed on P. That equation is easily verified to be (35.1) with both sides multiplied by $mu(u-2)^2$.

Finally, the values of $\dot P$ required in (c) are easily obtained by differentiating (35.1) at $u = 0$ or $u = 1$ and using the relations $P(0) = 0$, $P(1) = 1$.

36. Monotonicity intervals for P

The rational function P defined by (34.1) with a fixed integer $m \geqslant 2$ has a nonzero derivative at every $u \in \mathbf{R}$ except z, z^* (for even m), or $u_\pm, u_\pm^*$ (for odd m), with $u^* = u/(u-1)$ if $u \ne 1$ (cf. (20.2)), z as in (21.4), and $u_\pm$ described below. The values/limits of P at selected points, along with its strict monotonicity types on the intervening intervals, marked by slanted arrows, are listed below, with z, w, x as in (21.4) – (21.5) and Lemma 25.2(b). First, for even m,

value or limit at:	0		z^*		1		2
for P (m even):	0	$\searrow$	$-\infty\uparrow+\infty$	$\searrow$	1	$\searrow$	0

(36.1)

If m is odd, P restricted to $[0,2]$ reaches its extrema at unique points $u_\pm$ with

$$0 < u_- < w^* < z^* < x^* < u_+ < 1 \quad \text{and} \quad -1/m < P(u_-) < 0 < 1 < P(u_+). \tag{36.2}$$

(See also (vi) in §45.) Here is the corresponding diagram:

value at:	0		u_-		w^*		z^*		x^*		u_+		1		2
of P (m odd):	0	$\searrow$	$P(u_-)$	$\nearrow$	0	$\nearrow$	$z/(z-2)$	$\nearrow$	q_*	$\nearrow$	$P(u_+)$	$\searrow$	1	$\searrow$	0

(36.3)

Remark 36.1. In view of (d), (a), (b) of §35 and Lemma 20.1, the monotonicity intervals of P on the *whole* real line can be easily determined using (36.1) – (36.3). Specifically, P always decreases from 0 to -1 on $[2,\infty)$, that is, forms a decreasing diffeomorphism $[2,\infty) \to (-1,0]$. Similarly, when m is even, P decreases on $(-\infty, z)$ (or, on $(z, 0]$) from -1 to $-\infty$ (or, respectively, from ∞ to 0). Finally, if m is odd, P decreases on $(-\infty, u_+^*]$ from -1 to $P(u_+^*) = -P(u_+)$, increases on $[u_+^*, w]$ from $-P(u_+)$ to 0, and then continues increasing on $[w, u_-^*]$, from 0 to $P(u_-^*) = -P(u_-)$, while on $[u_-^*, 0]$ it decreases from $-P(u_-)$ to 0.

37. Proofs of the above claims

For any $u \ne 0$ the right-hand side of (35.1) is a quadratic polynomial in P having real roots $P_\pm(u)$ with $P_-(u) < P_+(u)$. Clearly, $u \mapsto P_\pm(u)$ are real-analytic functions on $(-\infty,0) \cup (0,\infty)$. As shown below, for $\dot P_\pm = dP_\pm/du$ and (in (iii), (iv)) for any $u \in \mathbf{R} \smallsetminus \{0,1\}$ at which $P(u)$ is defined, cf. (a), (b) in §35,

i) $P_- < 0 < P_+$, (ii) $\dot P_\pm < 0$, (iii) $(u-1)\dot P(u) < 0$ if and only if $P_-(u) < P(u) < P_+(u)$.

iv) $\dot P(u) = 0$ if and only if $P(u) = P_-(u)$ or $P(u) = P_+(u)$.

v) $P_\mp(0^\pm) = 0$ and $\dot P_\mp(0^\pm) = [2(1-m)]^{-1}$ (one-sided limits at 0),

vi) $P_+(1) = 1$, while $P_-(1) = -1/m$ and $\dot P_+(1) = -2(m-1)/(m+1)$.

In addition to (v), P_+ on $(-\infty,0)$ and P_- on $(0,\infty)$ are restrictions of a single analytic function on $\mathbf{R}$ with the value 0 at 0. The strict-monotonicity intervals (marked by slanted arrows) and some limits of $P_\pm$ appear in the diagram

$$\begin{array}{lccccccc}
\text{value or limit at:} & -\infty & & 0 & & 1 & & +\infty \\
\hline
\text{for } P_+ \text{ (any } m\text{):} & 1/m & \searrow & 0\uparrow+\infty & \searrow & 1 & \searrow & 1/m \\
\text{for } P_- \text{ (any } m\text{):} & -1 & \searrow & -\infty\uparrow 0 & \searrow & -1/m & \searrow & -1
\end{array} \tag{37.1}$$

In fact, (i) is obvious as $P_-(u)P_+(u) = -1/m < 0$, while (35.1) gives (iii), (iv). Next, equating the right-hand side of (35.1) to zero we obtain $(P+1)(mP-1)u = 2(m-1)P$. This defines a set in the uP-plane, namely, the union of $\{(0,0)\}$ and the graphs of $P_\pm$, which, at the same time, forms the graph of the rational function u of the variable P with $u = 2(m-1)P/[(P+1)(mP-1)]$. The latter function has a negative derivative except at the two poles $P=-1$ and $P=1/m$, and tends to 0 as $P\to\pm\infty$, so that (ii), (v), (vi) and (37.1) follow easily.

Next, $P(0), P(1), P(2)$ are given by (c) in §35, and, if m is even, $P(u)\to\pm\infty$ as $u\to z^\pm$ by (21.4) and (34.1), so that (d) in §35 gives $P(u)\to\pm\infty$ as $u-z^*\to 0^\pm$. If m is odd, $1+\varSigma(0)/G(z) = 0$ (see (21.5), (5.6.ii)), and so $P(z^*) = z/(2-z)$ by (34.1). To prove (36.1) – (36.3), we now only need to show that $\dot P<0$ on $(0,z^*)\cup(z^*,1)\cup(1,2]$ for even m, while, for odd m, there exist $u_\pm\in\mathbf{R}$ with (36.2) such that $\dot P<0$ on $(0,u_-)\cup(u_+,1)\cup(1,2]$ and $\dot P>0$ on (u_-,u_+). (Note that, according to (c) in §35, $\dot P<0$ at 0 and 1.)

By using (v) – (vi) above, (37.1) and (c) in §35 to find the value of $\varPhi = P-P_\pm$ at 1 or its right-sided limit at 0 (and the same for $\dot\varPhi$, as needed), we see that $P-P_+$ changes sign at 1, from $+$ to $-$, while $P-P_->0$ at 1 and $P-P_\pm<0$ at every $u>0$ close to 0. Also, $P-P_+<0<P-P_-$ at 2 by (i), since (c) in §35 states that $P(2)=0$.

In view of (ii), (iv) above and (a), (b) in §35, the assumptions of Remark 18.1 are satisfied by $\varPhi = P-P_\pm$ on $\mathcal{I}=(0,1)$ (for odd m), as well as on $\mathcal{I}=(0,z^*)$ or $\mathcal{I}=(z^*,1)$ (for even m), and on $\mathcal{I}=(1,2]$ (for all m). In each case, the inequalities of the last paragraph lead, as shown below, to a unique choice between the two alternatives allowed in the conclusion of Remark 18.1.

First, as $P-P_+$ (or, $P-P_-$) restricted to $(1,2]$ is negative (or, respectively, positive) near both endpoints, Remark 18.1 implies that $\operatorname{sgn}(P-P_\pm)$ is constant on $(1,2]$, and hence $P_-<P<P_+$ on $(1,2]$. By (iii), this yields $\dot P<0$ on $(1,2]$.

Secondly, as $P-P_\pm$ restricted to $(0,1)$ is negative near the endpoint 0 and positive near the endpoint 1, Remark 18.1 gives rise to two different cases, depending on m. If m is even, the infinite limits of P at z^*, already verified to be those required in (36.1), show that $P-P_\pm$ on $\mathcal{I}=(0,z^*)$ (or, $\mathcal{I}=(z^*,1)$) is negative (or, respectively, positive) near both endpoints, and hence, by Remark 18.1, it is so everywhere in $\mathcal{I}$. Since this applies to both signs $\pm$, (iii) and (iv) give $\dot P<0$, for even m, both on $(0,z^*)$ and $(z^*,1)$. If m is odd, however, $P-P_\pm$ is of class C^1 everywhere in $(0,1)$, and so Remark 18.1 implies the existence of unique points $u_\pm\in(0,1)$ such that $\operatorname{sgn}(P-P_\pm)$ at any $u\in(0,1)$ equals $\operatorname{sgn}(u-u_\pm)$. Hence, by (i), $P=P_+>P_-$ at u_+, and so $\operatorname{sgn}(u_+-u_-)=1$, that is, $u_-<u_+$. Now, by (iii), (iv) and the last paragraph, $\dot P<0$ on $(0,u_-)\cup(u_+,2]$ and $\dot P>0$ on (u_-,u_+). Thus, as u increases from 0 to u_-, then to u_+, then to 1, and finally to 2, the value of P decreases from 0 to $P(u_-)$, then increases to $P(u_+)$, then decreases to 1 and after that continues decreasing to 0. Any point $u\in(0,1)$ with $0\leqslant P(u)\leqslant 1$ must therefore lie in (u_-,u_+). This includes $u=w^*$ and $u=x^*$, as $P(w^*)=0$ (see (a) in §35) and $P(x^*) = -q_*\in(0,1)$ (by (d) in §35 and Remark 34.2 as $P(x)$ is the value of p at (x,x^*), cf. §34). We have thus proved (36.1), (36.3) and the first part of (36.2), since $w^*<z^*<x^*$ by Lemma 20.1 with $x<z<w<0$ (cf. the lines preceding (21.5)). The description just given of the monotonicity intervals of P on $[0,2]$ also shows that P assumes its extrema in $[0,2]$ at u_- and u_+, while $P(u_-)<0<1<P(u_+)$. Also, $P(u_-)>-1/m$ for odd m, since the minimum

$P(u_-)$ equals, by (iv), the value at u_- of P_- (not of P_+, as $P_- < P_+$ by (i)), and so, by (37.1), $P(u_-) = P_-(u_-) > P_-(1) = -1/m$.

38. The values assumed by p on the moduli curve

Let $m \geqslant 2$ be a fixed integer. The restrictions to the connected components of the moduli curve $\mathcal{C}$ (see Corollary 33.2) of the functions δ and p defined in §1 and §34 have the following properties, which, as explained below, are easy consequences of the results of the preceding sections. In the case of δ, (a) – (e) simply repeat its definition from §1, to provide a convenient reference.

a) On the $\mathcal{T}$-beam $\mathsf{X} \cap \mathcal{T}$ of X, when m is odd: $\delta = 1$ and the range of p is $(-1, q_*]$, for $q_* \in (-1, 0)$ as in (27.1). Every value in $(-1, q_*)$ is assumed by p exactly twice in $\mathsf{X} \cap \mathcal{T}$, while q_* is assumed just once, at (x, x^*). Two different points of the $\mathsf{X} \cap \mathcal{T}$ have the same value of p if and only if they are each other's images under the involution (28.1).
b) On the $\mathcal{H}$-beam $\mathsf{X} \cap \mathcal{H}$, when m is odd: $\delta = 1$, the range of p is $[P(u_+^*), z/(2-z))$, while $P(u_+^*) < -1 < z/(2-z) < 0$ by (36.2) with (d), (a) in §35, and as $z < 0$ (see §21). Cf. (iii) in §45. The values in $(P(u_+^*), -1)$ are assumed by p in $\mathsf{X} \cap \mathcal{H}$ twice, those in the union $\{P(u_+^*)\} \cup [-1, z/(2-z))$ just once.
c) On $\mathsf{!}$, when m is odd: $\delta = -1$ and the range of p is $[0, P(u_-^*)]$, with $0 < P(u_-^*) < 1/m$. Every value in $[0, P(u_-^*)]$ is assumed by p twice in $\mathsf{!}$, except for 0 and $P(u_-^*)$, assumed just once, at $(0,0)$ and (u_-^*, u_-).
d) On $\mathsf{!}$, when m is even, (z, z^*) is the only point at which $\delta = 0$, while $\delta = 1$ (or, $\delta = -1$) on the subset of $\mathsf{!}$ formed by all $(u, u^*) \in \mathsf{!}$ with $u < z$ (or, respectively, $z < u \leqslant 0$); that subset is mapped by p bijectively onto $(-\infty, -1)$ (or, respectively, $[0, \infty)$).
e) On I, for every m, we have $\delta = 1$ and $p : \mathsf{I} \to (0, 1)$ is bijective.

In fact, (a) is obvious from the last sentence in Proposition 29.1(ii), along with the easily-verified invariance of q under (28.1) and Remark 34.2.

Moreover, $v = u^*$ for every point (u, v) of the moduli curve that does not lie in the $\mathcal{T}$-beam $\mathsf{X} \cap \mathcal{T}$ (see §33). Now (b) – (e) are immediate, since p at (u, v) then equals $P(u)$, for the function P defined by (34.1), which has the limits/values and monotonicity intervals are described in (36.1) and (36.3). The inequalities in (b), (c) easily follow from (36.2) and (d) in §35.

Proof of Theorem 1.6. Assertions (i), (ii) are obvious from Definition 1.1 and (c), (d) above; that the set of p-rational points in $\mathsf{!}$ is countably infinite for odd m as well follows from (c), as $[0, P(u_-^*)] \cap \mathbf{Q}$ is infinite. ☐

39. More on p-rationality

Let $\mathcal{C}$ again denote the moduli curve for a fixed integer $m \geqslant 2$ (see §33). In §1 and §34 we introduced two functions on $\mathcal{C}$, namely, $\delta : \mathcal{C} \to \{-1, 0, 1\}$ and the rational function p. We also observed that p, declared to be 0 at $(0,0) \in \mathcal{C}$, is defined everywhere in $\mathcal{C}$, except at (z, z^*) when m is even. Both functions are involved in Definition 1.1, which describes a subset of the uv-plane $\mathbf{R}^2$, contained in $\mathcal{C}$, and consisting of what we call the p-rational points.

Remark 39.1. For a fixed integer $m \geqslant 2$, a point $(u, v) \in \mathbf{R}^2$ with $u < v$ is p-rational if and only if it can be used to construct a quadruple (M, g, m, τ) with (0.1) or (0.2) as described in §1. This is in turn equivalent to the existence of objects required in (3.1) – (3.2) for the given u, v and the function $Q \in \mathbf{V}$, unique up to a factor (cf. Lemma 9.1), which satisfies (2.5) on $I = [u, v]$ and is positive on the interior of I.

Since the objects with (3.1) always exist, while the existence of those in (3.2) is equivalent to (4.2), the above assertion will follow once we show that the constants δ, p in (4.2), defined by (4.1), coincide with the values at (u,v) of the functions δ and p defined in §34 and §1. This is obvious for p when A in (2.3) is nonzero (see the line preceding (5.7)), and for both p,δ if $A=0$ (as $(u,v)\in\mathcal{C}\smallsetminus\{(0,0)\}$ due to the definition of $\mathcal{C}$, and hence, according to §34, m then is even and $(u,v)=(z,z^*)$, so that $\delta=0$ while p is undefined, for either meaning of δ and p). Finally, when $A\neq 0$, the inequality in (3.1) gives, for δ as in (4.1.ii), $\delta=\operatorname{sgn}\varphi(t)$ whenever $u<t<v$, with $\varphi(t)=(t-1)AQ(t)$. As $u\neq 1$ (cf. Remark 19.3) and $\varphi(u)=0\neq\dot\varphi(u)$ unless $A=0$ (see (2.5)), this yields $\delta=\operatorname{sgn}\dot\varphi(u)$, that is, $\delta=\operatorname{sgn}[(u-1)A\dot Q(u)]$, where $(\ )^{\cdot}=d/dt$. Hence, by (4.1.i), $\delta=\operatorname{sgn}[(u-1)p]$, with either definition of p at (u,v). (We already showed that both definitions agree.) This last formula for δ is clearly consistent with (a) – (e) of §38: $u<0$ and $p<0$ on both beams of X, while on $\mathsf{!}$ we have $u<0$ and p is represented by P with (36.1) – (36.3), and, finally, $u>1$ and $p>0$ on I.

We will now describe some results of [11] and use them to prove Theorem 1.3. First, according to the discussion following [11, Proposition 33.1 in §33],

i) Any quadruple (M,g,m,τ) satisfying (0.1) or (0.2) must belong to one of four disjoint *types* (a), (b), (c1), (c2) defined in §33 of [11].

ii) Type (b) cannot occur, as it contradicts the compactness assumption made in (0.1) and (0.2). (In §33 of [11] compactness of M is not assumed.)

iii) Quadruples of type (a) (or, (c1)) all arise from the construction in §43 (or, §3) of this paper.

Namely, (ii) – (iii) follow from [11, Theorems 33.2, 33.3, 34.3 and Remark 2.4].

Lemma 39.2. *One of the following two assertions holds for any given quadruple (M,g,m,τ) with* (0.1) *or* (0.2).

($*$) *Up to a τ-preserving biholomorphic isometry, (M,g,m,τ) arises from the construction in* §43*, or from that in* §3.

($**$) *Conditions* (2.5) *in* §2 *are satisfied by some nontrivial closed interval I with $1\in I$ and some $Q\in\mathbf{V}$, with $\mathbf{V}$ as in* (2.4) *for this given m.*

In fact, by (i) – (iii) above, type (b) is excluded, types (a) and (c1) lead to ($*$), while, for type (c2), Corollary 35.1 in [11] yields ($**$).

Proof of Theorem 1.3. Case ($**$) in Lemma 39.2 is made impossible by Proposition 19.2, while the construction in §3 amounts to that described before Proposition 1.2: for $I=[u,v]$ used in §3, p-rationality of (u,v) is obvious from Remark 39.1 . $\square$

40. Bounds on z and x

Let $R_m(t)=-G_m(t)/\Sigma_m(0)$, with F_m,E_m,Σ_m,G_m standing for F,E,Σ,G, as in (5.2). Also, let $\Sigma_0(0)=-1/2$ and $R_0(t)=1$. Then, for any integer $m\geqslant 1$,

$$\begin{array}{rl} \text{i)} & \Sigma_m(0) = (4-6/m)\,\Sigma_{m-1}(0)\,, \qquad \Sigma_1(0) = 1\,,\\ \text{ii)} & R_m(t)=1-m\sigma R_{m-1}(t)/(4m-6), \quad R_1(t)=1+\sigma/2, \text{ with } \sigma=t^2/(1-t),\\ \text{iii)} & (R_1(-2),R_2(-2),R_3(-2),R_4(-2))=(5/3,-11/9,49/27,13/405),\\ \text{iv)} & \Sigma_m(0)R_m(t) = \Sigma_m(0) - \Sigma_{m-1}(0)\sigma + \ldots + (-1)^m\Sigma_0(0)\sigma^m, \end{array} \tag{40.1}$$

whenever $t\in(-\infty,0)$, where $\sigma=t^2/(1-t)$ in (iv) as well. Namely, (5.6.i) gives (i), and (ii) follows since (5.2.ii) clearly remains valid, for $t\neq 1$, even if one replaces E_m by $G_m=E_m-F_m/2$. Finally, (i) and (ii) easily imply (iii) and (iv).

Any given $\sigma > 0$ corresponds as in (40.1) to a unique $t < 0$. In fact, since $\sigma = -t-1+1/(1-t)$, we have $d\sigma/dt < 0$, and taking the limits of σ we see that

$$(-\infty, 0) \ni t \,\mapsto\, \sigma = t^2/(1-t) \in (0,\infty) \quad \text{is a decreasing diffeomorphism.} \tag{40.2}$$

With $\varSigma_m(0)$ again denoting the sequence given by (40.1.i) (or (5.6.i)),

$$\begin{array}{ll} \text{a)} & a_m \,=\, m^{3/2}4^{1-m}\varSigma_m(0) \;\text{ is a positive decreasing function of }\; m \geqslant 1, \\ \text{b)} & a_m \,\to\, 1/\sqrt{\pi} \;\text{ as }\; m \to \infty, \\ \text{c)} & \varSigma_m(0) \,=\, (4-6/2)(4-6/3)\ldots(4-6/m) \;\text{ for any integer }\; m \geqslant 2. \end{array} \tag{40.3}$$

Namely, (40.1.i) yields both (c) and $4(a_m/a_{m-1})^2 \,=\, m(2m-3)^2/(m-1)^3 \,<\, 4$ for $m \geqslant 2$, which gives (a), and Wallis's formula $\lim_{m\to\infty} 2m^{1/2}\prod_{j=1}^{m-1}[2j/(2j+1)] = \sqrt{\pi}$ implies (b) since $1/\prod_{j=1}^{m-1}[2j/(2j+1)] = 2^{1-m}[(m-1)!]^{-1}\prod_{j=1}^{m-1}(2j+1) = 4^{1-m}[(m-1)!]^{-2}(2m-1)!$, which, by (5.6.i), equals $4^{1-m}(2m-1)m\varSigma_m(0) = (2m-1)m^{-1/2}a_m$.

For a real variable r and for $\varSigma_m(0)$ as in (40.1.i) with $\varSigma_0(0) = -1/2$, let

$$\varPsi_m(r) = \varSigma_0(0) - \varSigma_1(0)r + \ldots + (-1)^m\varSigma_m(0)r^m, \quad \varPsi_\infty(r) = -\,\frac{\sqrt{1+4r}}{2}\,, \tag{40.4}$$

with $r \geqslant -1/4$ in $\varPsi_\infty(r)$. By (40.1.i), $\varPsi_m$ satisfies the initial value problem

$$(2r+1/2)d\varPsi_m/dr \,=\, \varPsi_m(r) \,+\, (2m-1)(-1)^m\varSigma_m(0)r^m, \;\; \varPsi(0) = -1/2, \tag{40.5}$$

while, by (40.1.ii), the power series whose partial sum appears in (40.4) has the convergence radius $1/4$. The sum of this series is $\varPsi_\infty(r)$, as one sees either noting that the sum satisfies on $(-1/4, 1/4)$ the initial value problem $(1+4r)d\varPsi_\infty/dr = 2\varPsi_\infty(r)$ with $\varPsi(0) = -1/2$ (which one may derive from (40.5) and (40.1.i), as well as directly from (40.1.i)), or using (40.1.iv) to verify that $(-1)^m\varSigma_m(0)$ equals, for every $m \geqslant 0$, the mth Taylor coefficient of $\varPsi_\infty(r)$ at $r = 0$. Also, by (40.1.i),

$$\varPsi_m \,\to\, \varPsi_\infty \;\text{ as }\; m\to\infty, \;\text{ uniformly on }\; [-1/4, 1/4], \tag{40.6}$$

since $4|\varSigma_m(0)r^m| \leqslant m^{-3/2}a_m$ whenever $|r| \leqslant 1/4$.

If m is odd and $m \geqslant 3$, since $\varSigma_{2k-1}(0) = k\varSigma_{2k}(0)/(4k-3)$ by (40.1.i), we get

$$\varPsi_{m-1}(r) = -\frac{1}{2} + \sum_{k=1}^{(m-1)/2}\varXi_k(r), \;\text{ where } \varXi_k(r) = \varSigma_{2k}(0)r^{2k-1}\left[r - \frac{k}{4k-3}\right], \tag{40.7}$$

by grouping terms in (40.4). Obviously, $\varXi_k \leqslant 0$ on the interval $[0, k/(4k-3)]$ and, since these intervals form a descending sequence, $\varXi_k \leqslant 0$ on $[0,(m-1)/(4m-10)]$ for all $k = 1,\ldots,(m-1)/2$. Thus, by (40.7),

$$\varPsi_{m-1}(r) \leqslant -1/2 \text{ whenever } m \geqslant 3 \text{ is odd and } 0 \leqslant r \leqslant (m-1)/(4m-10). \tag{40.8}$$

Remark 40.1. Let $R_m(t)$ and χ_m be defined by (40.1) and $\chi_m = x^2/(1-x)$, with $x < 0$ as in Lemma 25.2(b) for any $t \in (-\infty, 0)$ and an odd integer $m \geqslant 3$. Setting $\sigma = t^2/(1-t)$, we have

a) if $R_m(t) \leqslant 1$, then $\chi_m > \sigma$,

b) $\operatorname{sgn}(\chi_m - \sigma) = \operatorname{sgn}[R_{m-1}(t) + K_m(\sigma)]$, where $K_m(\sigma) = 2(2m-3)/[(m-1)\sigma + 2(2m-1)]$.

In fact, $S(t)/[\varSigma(0)(1-t)] = [(m-1)\sigma + 2(2m-1)][1 - R_m(t)] + m\sigma$ as $t^2 = (1-t)\sigma$, with S as in (24.5.b). Since $m, \sigma, 1-t$ and $\varSigma(0)$ are positive (cf. (2.2)), this, combined with Lemma 25.2(c) and (40.2), gives (a) and, due to (40.1.ii), also (b).

Lemma 40.2. *Given an odd integer* $m \geqslant 3$, *let* $\zeta_m = z^2/(1-z)$ *and* $\chi_m = x^2/(1-x)$ *for* $z, x \in (-\infty, 0)$ *defined in* (21.5) *and* Lemma 25.2(b). *Then*

i) $0 < 4 - 34/(m^{1/4}+8) < \zeta_m < \chi_m \leqslant 4 - 6/(m-1) < 4$, *with all inequalities strict if* $m \geqslant 5$,

ii) ζ_m *and* χ_m *are strictly increasing functions of* m,

iii) $\chi_m \to 4$ *and* $\zeta_m \to 4$ *as* $m \to \infty$.

Proof. Let $m \geqslant 3$ be odd. For $r \in (0,\infty)$ and Ψ_{m-1}, sgn as in (40.4) and §4,

$$\begin{array}{ll}\text{a)} & \mathrm{sgn}\,(r - 1/\chi_m) \,=\, \mathrm{sgn}\,\Delta_m(r), \quad \text{b)}\ \mathrm{sgn}\,(r-1/\zeta_m) \,=\, \mathrm{sgn}\,\Psi_{m-1}(r), \\ & \text{where} \quad \Delta_m(r) \,=\, \Psi_{m-1}(r) \,+\, m\Sigma_m(0)r^m/[2(2m-1)r+m-1].\end{array} \tag{40.9}$$

In fact, $\Delta_m(r) = \sigma^{1-m}\Sigma_{m-1}(0)[R_{m-1}(t) + K_m(\sigma)]$ for $\sigma = 1/r$, by (40.1.iv) (with $m-1$ rather than m) and (40.1.i), and so a) is clear from Remark 40.1(b), Lemma 25.2(c) and (40.2). Similarly, (b) follows as $\mathrm{sgn}\,(t-z)$ for any $t<0$ equals $\mathrm{sgn}\,(\zeta_m - \sigma) = \mathrm{sgn}\,(r - 1/\zeta_m)$, where $r = 1/\sigma$ with σ as in (40.2); on the other hand, by (21.5), $\mathrm{sgn}\,(t-z)$ also coincides with $\mathrm{sgn}\,[G(t)-E(0)]$, that is, $\mathrm{sgn}\,[G(t)+\Sigma_m(0)]$ (cf. (5.6.ii)), and hence equals $\mathrm{sgn}\,[\Sigma_m(0) - \Sigma_m(0)R_m(t)]$ (see the definition of R_m, preceding (40.1)), while $\Sigma_m(0) - \Sigma_m(0)R_m(t) = \sigma^m\Psi_{m-1}(1/\sigma)$ by (40.1.iv) and (40.4).

Next, $\zeta_m > 0$ as $z<0$, while $\zeta_m < \chi_m$ since $x<z$ (Lemma 25.2(b)) or, equivalently, by (40.9) for $r = 1/\zeta_m$ (as $\Delta_m(r) > \Psi_{m-1}(r)$). The lower bound on ζ_m in (i) is obtained by finding (as described below) a constant $\alpha > 0$ and a positive increasing function Y_m of the odd integer $m \geqslant 3$ such that $\alpha\Psi_{m-1}(1/4)$ and $\alpha\Psi'_{m-1}(1/4)$ are both greater than -1, while $\alpha\Psi''_{m-1}(r) > 2Y_m$ for every odd $m \geqslant 3$ and every $r \in [1/4,\infty)$, with $(\ )' = d/dr$. Then, clearly, $\alpha\Psi_{m-1}(r) > f_m(r)$ at every $r \geqslant 1/4$, where $f_m(r) = Y_m(r-1/4)^2 - (r-1/4) - 1$ is the quadratic function of r whose value, derivative and second derivative at $r = 1/4$ are $-1,-1$ and $2Y_m$. By (40.9.b), this gives $1/r < \zeta_m$ for any $r \geqslant 1/4$ at which $f_m(r) > 0$. Next, $f_m(r) > 0$ for $r = (\beta+1)Y_m^{-1/2} + 1/4$, where $\beta = Y_3^{-1/2}$. Namely, $f_m(r) = \beta^2 + 2\beta - (\beta+1)Y_m^{-1/2} \geqslant \beta$ as $Y_m \geqslant Y_3$, that is, $Y_m^{-1/2} \leqslant \beta$. Thus, $\zeta_m > 1/r = 4 - 16(\beta+1)/[Y_m^{1/2} + 4(\beta+1)]$. If, in addition, $Y_m = 16\sqrt{m}/\gamma^2$ with a constant $\gamma > 0$ independent of m, this gives $\zeta_m > 4 - 4(\beta+1)\gamma/[m^{1/4} + (\beta+1)\gamma]$.

A choice of α and Y_m with the required properties is $\alpha = \sqrt{2}$ and $Y_m = \sqrt{2m/\pi}$, that is, $Y_m = 16\sqrt{m}/\gamma^2$ with $\gamma = (128\pi)^{1/4}$. In fact, the sequences $\Psi_m(1/4)$ and $\Psi'_m(1/4)$, with $m = 1,2,3,\dots$, converge to $-1/\sqrt{2}$, as one sees setting $r = 1/4$ in (40.6) and, respectively, (40.5) (where the term involving $\Sigma_m(0)$ tends to 0 by (40.3.a)). On the other hand, $\Psi_{m-1}(1/4)$ and $\Psi'_{m-1}(1/4)$ are decreasing functions of the *odd* integer $m \geqslant 3$, since $\Xi_k(r)$ and $\Xi'_k(r)$ are, by (40.7), negative for $r = 1/4$ and any $k \geqslant 1$. Also, $\Xi''_k(r)$ is a nondecreasing function of $r \in [1/4,\infty)$, as $\Xi''_k(r) = 2k(2k-1)\Sigma_{2k}(0)r^{2k-3}[r - (k-1)/(4k-3)]$. (Note that $r - (k-1)/(4k-3)$ is a *positive* increasing function of $r \geqslant 1/4$.) By (40.7), the same is true of $\Psi''_{m-1}(r)$. Thus, we just need to establish the inequality $\sqrt{2}\,\Psi''_{m-1}(1/4) > 2Y_m$ for $Y_m = \sqrt{2m/\pi}$. Now (40.3) and the above formula for $\Xi''_k(r)$ give $\Xi''_k(1/4) = 2a_{2k-1}/\sqrt{2k-1} > 2/\sqrt{(2k-1)\pi}$ for all $k \geqslant 1$, and so, by (40.7), $\Psi''_{m-1}(1/4) > \sqrt{4m/\pi}$, as required. (In fact, easy induction gives $1^{-1/2} + 3^{-1/2} + 5^{-1/2} + \ldots + (m-2)^{-1/2} \geqslant \sqrt{m}$ for any odd integer $m \geqslant 3$, as $(m^{1/2} + m^{-1/2})^2 > m+2$.) Now $(\beta+1)\gamma = (8\pi/\sqrt{3})^{1/2} + (128\pi)^{1/4} \approx 8.29$, for $\beta = Y_3^{-1/2}$, so that $(\beta+1)\gamma > 8$ and $4(\beta+1)\gamma < 34$. The inequality concluding the last paragraph thus gives the lower bound for ζ_m appearing in (i).

To prove the remainder of (i) we may assume that $m \geqslant 5$, since Example 27.4 gives $x = -(\sqrt{5}+1)/2$ for $m = 3$ and hence $\chi_3 = 1 = 4 - 6/2$. The inequality $\chi_m < 4 - 6/(m-1)$ will be obvious from (40.9.a) once we show that

$$m\Sigma_m(0)r^m/(m-1) \,<\, 1 \quad \text{for } m \geqslant 5 \text{ and } r = (m-1)/(4m-10), \tag{40.10}$$

since the definition of Δ_m in (40.9) combined with (40.8) then will give $\Delta_m(r) < 0$. Note that $1/[2(2m-1)r+m-1] < 1/[2(m-1)]$.

We derive (40.10) from the fact that $(4r)^{m-5/2} = (1+1/\nu)^{3\nu/2} < e^{3/2} < 5$, for m, r as in (40.10), with $\nu = (2m-5)/3$. (That $\nu\log(1+1/\nu)$, and hence also $(1+1/\nu)^\nu$, is an increasing function of $\nu > 1$, is clear as $\mu^{-1}\log(1+\mu)$ is a decreasing function of $\mu = 1/\nu \in (0,1)$, which follows since

$(1+\mu)\mu^2$ times its derivative decreases on $[0,1]$ from 0 to $\log(e/4)$.) Thus, $(4r)^m < 5(4r)^{5/2} < 20$, as $r \leqslant 2/5$. Also, (40.3) for $m \geqslant 5$ gives $a_m < a_2 = \sqrt{2}/8 < 1/4$ and $\varSigma_m(0) < m^{-3/2}4^{m-1}$, so that $m\varSigma_m(0)r^m/(m-1) < m^{-1/2}(4r)^m/[4(m-1)]$, which is less than $5m^{-1/2}/(m-1) < 1$. This yields (i) and, consequently, (iii).

To prove (ii) for ζ_m, let us fix $m \geqslant 5$ and set $r = 1/\zeta_m$. By (40.7) and (40.9.b), $-\varPsi_{m-3}(r) = \varPsi_{m-1}(r) - \varPsi_{m-3}(r) = \varSigma_{m-1}(0)r^{m-2}[r - (m-1)/(4m-10)]$, and so $\varPsi_{m-3}(r) < 0$ since (i) gives $\zeta_m < 4 - 6/(m-1)$, that is, $r > (m-1)/(4m-10)$. Now (40.9.b), for $m-2$ rather than m, gives $\zeta_m = 1/r > \zeta_{m-2}$.

Finally, to obtain (ii) for χ_m, note that, for any $r > 0$ and $m \geqslant 5$,

$$\frac{\varDelta_m(r) - \varDelta_{m-2}(r)}{(2m-5)(m-1)\varSigma_{m-1}(0)} = \frac{r^{m-2}(1+4r)^2[r - (m-1)/(4m-10)]}{[(4m-2)r+m-1][(4m-10)r+m-3]}, \qquad (40.11)$$

since (40.7) leads to a massive cancellation of terms in $\varDelta_m(r) - \varDelta_{m-2}(r)$ and, by (40.1.i), $\varSigma_{m-2}(0) = (m-1)\varSigma_{m-1}(0)/(4m-10)$, $\varSigma_m(0) = (4-6/m)\varSigma_{m-1}(0)$. Applied to $r = 1/\chi_m$, this gives, by (40.9.a), $-\varDelta_{m-2}(r) = \varDelta_m(r) - \varDelta_{m-2}(r) > 0$, as the inequality $\chi_m < 4 - 6/(m-1)$ in (i) amounts to $r > (m-1)/(4m-10)$. Thus, $\varDelta_{m-2}(r) < 0$ and (40.9.a) with m replaced by $m-2$ yields $r < 1/\chi_{m-2}$, that is, $\chi_m > \chi_{m-2}$, completing the proof. □

Remark 40.3. Let $x_\infty = -2(1+\sqrt{2}) \approx -4.83$. Combined with (40.2), Lemma 40.2 immediately implies that both z and x are strictly decreasing functions of the odd integer $m \geqslant 3$, such that $z > x > x_\infty$, while $x \to x_\infty$ and $z \to x_\infty$ as $m \to \infty$.

41. Further inequalities

Let $q_* \in (-1,0)$ depend on an odd integer $m \geqslant 3$ as in (27.1), and let $\ell_\infty = 9 + 6\sqrt{2}$. Then, with $\ell_m \in (1,\infty)$ such that $q_* = -1 + 1/\ell_m$,

a) $\ell_m < 18$ for every odd $m \geqslant 3$. More precisely, $\ell_m < \ell_\infty \approx 17.485$.
b) $\ell_m < m$ whenever m is odd and $m \geqslant 9$.
c) $\ell_m > m$ if $m \in \{3,5,7\}$, while $8 < \ell_9 < 9 < \ell_{11} < 10$.
d) $\ell_3 = (9 + 3\sqrt{5})/4 \approx 3.927$,
e) ℓ_m is a strictly increasing function of m and $\ell_m \to \ell_\infty$ as $m \to \infty$.

In fact, by (27.1), $\ell_m = \xi(\xi + \sqrt{\xi^2-1})$ for $\xi = \xi_m$ given by $\xi = 1 + \chi/2$ with $\chi = \chi_m$ as in Lemma 40.2. Thus, if $m = 3$, Example 27.4 gives $x = -(\sqrt{5}+1)/2$, so that $\chi_3 = 1$ and $\xi_3 = 3/2$, which yields (d), and hence (c) for $m = 3$.

Obviously, ℓ_m is an increasing function of $\xi_m > 1$, and hence of $\chi_m > 0$. This has several consequences. First: (e), (a) are obvious from Lemma 40.2, and (a) gives (b) for $m \geqslant 19$. Next, $R_4(-2) > 0$ in (40.1.iii), and so $R_5(-2) < 1$ by (40.1.ii); hence $\chi_5 > 4/3$, from Remark 40.1(a) for $t = -2$, so that $\xi_5 > 5/3$, and (c) for $m = 5$ follows. Finally, (b) and (c) for $m = 7, 9, \dots, 17$ are, similarly, numerical consequences of the bounds on χ_m provided by (48.1) below.

Remark 41.1. Lemma 40.2(i) clearly implies some explicit, though complicated bounds for ℓ_m. Replacing them with weaker but simpler estimates, we get $1 - 17/(m^{1/4}+8) < \ell_m/\ell_\infty < 1 - 1/m$. (The lower bound is of interest only when it is positive, that is, for very large m.) Namely, as $0 < \chi < 4 - 6/m$ for $\chi = \chi_m$, setting $\xi = 1 + \chi/2$ we get $\xi < 3 - 3/m$, and so $\sqrt{\xi^2-1} < 2\sqrt{2}\xi/3$ as $1 < \xi < 3$. Thus, $\ell_m < \ell_\infty\xi^2/9 < \ell_\infty\xi/3 < (1-1/m)\ell_\infty$, since $\ell_m = \xi(\xi + \sqrt{\xi^2-1})$ (see above). Similarly, for $L(\eta) = \eta(\eta + \sqrt{\eta^2-1}) - \ell_\infty(\eta - 2)$, with $\eta \in (1,\infty)$, we have $L(3) = 0$ and $dL/d\eta < 0$ at $\eta = 3$, while $d^2L/d\eta^2 > 0$ whenever $\eta > 2/\sqrt{3}$, as $d^2L/d\eta^2 = (2-\phi)(1+\phi)^2$ with $\phi = \eta/\sqrt{\eta^2-1}$, so that $d\phi/d\eta < 0$ and condition $\eta > 2/\sqrt{3}$ is equivalent to $\phi < 2$.

Therefore, $dL/d\eta < 0$, and hence $L(\eta) > 0$ for all $\eta \in [2/\sqrt{3}\,, 3)$, and, with $\eta = 3 - 17/(m^{1/4}+8)$, Lemma 40.2(i) now leads to our lower bound.

42. Some simple facts from number theory

The following lemma is a variation on the $s = 2$ case of the well-known fact that, for any integer $s \geqslant 2$, the probability that s randomly chosen positive integers have a common divisor other than 1 equals $1/\zeta(s)$, where ζ is the Riemann zeta function.

We allow ℓ_m and ℓ_∞ to be much more general here than in §41 or §46.

Lemma 42.1. *Given a real constant $\ell \geqslant 1$ and an integer $m \geqslant 1$, let $\mathcal{P}(\ell, m)$ be the set of all pairs (k, d) of relatively prime integers with $1 \leqslant k\ell < d \leqslant m$. Then*

a) $\pi^2|\mathcal{P}(\ell, m)|/m^2 \,\to\, 3/\ell$ *as $m \to \infty$, with $|\;|$ denoting cardinality,*
b) $\pi^2|\mathcal{P}(\ell_m, m)|/m^2 \,\to\, 3/\ell_\infty$ *as $m \to \infty$, whenever $\ell_\infty \in \mathbf{R}$ is the limit, as $m \to \infty$, of a function $m \mapsto \ell_m \in [1, \infty)$ defined on an infinite set of positive integers m.*

In fact, one obtains (a) by modifying a standard proof (see [8]) of Mertens's theorem. For the reader's convenience, we provide details in an appendix (§47).

Next, $\mathcal{P}(\ell', m) \subset \mathcal{P}(\ell, m)$ if $\ell' \geqslant \ell \geqslant 1$, and so, if $\ell' \geqslant \ell_m \geqslant \ell \geqslant 1$ for ℓ_m as in (b), then $|\mathcal{P}(\ell', m)|/m^2 \leqslant |\mathcal{P}(\ell_m, m)|/m^2 \leqslant |\mathcal{P}(\ell, m)|/m^2$. This obviously is the case for any fixed ℓ, ℓ' with either $\ell' > \ell_\infty > \ell \geqslant 1$ (when $\ell_\infty > 1$), or $\ell' > \ell_\infty = \ell = 1$, and sufficiently large m for which ℓ_m is defined. Taking the upper/lower limits, we now get (b) from (a) for ℓ, ℓ' arbitrarily close to ℓ_∞.

Remark 42.2. For $\mathcal{P}(\ell, m)$ as above, $(k, d) \mapsto k/d$ is a bijective correspondence between $\mathcal{P}(\ell, m)$ and the set of all those rational numbers in $(0, 1/\ell)$ which can be written as fractions with a denominator in $\{1, \ldots, m\}$. When $\ell = 1$, the elements of the latter set along with the numbers 0 and 1, listed in increasing order, form what is called the *Farey sequence of order m* (cf. [8]).

It is clear that the Farey sequence of order m has $2 + \varphi(2) + \ldots + \varphi(m)$ elements, where φ is the *Euler function*, assigning to a positive integer k the number of integers j such that $0 < j < k$ and j, k are relatively prime.

43. Examples with locally reducible metrics

This and the next three sections describe constructions of the four families, mentioned in §0, of quadruples (M, g, m, τ) satisfying (0.1) or (0.2).

The *first family* is represented by just one p-rational point $(0,0)$ on the moduli curve $\mathcal{C}$ (see the end of §2), and consists of those (M, g, m, τ) with (0.1) or (0.2) for which g is locally reducible as a Kähler metric. Such (M, g, m, τ) seem to be well-known, and can be constructed as follows.

Given an integer $m \geqslant 2$, real constants $\kappa < 0$ and $c \neq 0$, and a compact Kähler-Einstein manifold (N, h) of complex dimension $m - 1$ with the Ricci tensor $\mathrm{r}^{(h)} = \kappa h$, let $\mathcal{L}$ be any holomorphic line bundle over N carrying a fixed flat $\mathrm{U}(1)$ connection. Next, let $\mathcal{E} = N \times \mathbf{R}$ be the product real-line bundle over N with the obvious flat connection and Riemannian fibre metric, and let M be the unit-sphere bundle of the direct sum $\mathcal{L} \oplus \mathcal{E}$. Thus, M is a 2-sphere bundle over N. Since the direct-sum connection in $\mathcal{L} \oplus \mathcal{E}$ is flat and compatible with the direct-sum metric, its horizontal distribution is both integrable and tangent to the submanifold M, and so it gives rise to an integrable distribution which may also be called *horizontal*, and whose leaves, along with the $\mathbf{CP}^1$ fibres form, locally in M, the factor manifolds of a Cartesian-product decomposition.

Let g now be a metric on M such that the horizontal distribution is g-normal to the fibres and g restricted to it is the pullback of h under the bundle projection $M \to N$, while g on each fibre

equals $(3-2m)/\kappa$ times the standard unit-sphere metric. Thus, (M,g) is a Kähler manifold since, locally, it is a Riemannian product with the factors manifolds which are a 2-sphere S^2 of constant Gaussian curvature $\kappa/(3-2m)$ and (N,h), while the 2-sphere factors can be coherently oriented, which makes them Kähler manifolds of complex dimension one.

Finally, let $\tau : M \to \mathbf{R}$ be the composite $M \to \mathcal{L} \oplus \mathcal{E} \to \mathcal{E} \to \mathbf{R} \to \mathbf{R}$ of the inclusion mapping of M, followed by the direct-sum projection morphism, followed by the Cartesian-product projection $\mathcal{E} = N \times \mathbf{R} \to \mathbf{R}$, followed by the multiplication by the nonzero constant c in $\mathbf{R}$. In terms of a local Riemannian-product decomposition just described, with the S^2 factor treated as the sphere of radius $\sqrt{(3-2m)/\kappa}$ about 0 in a Euclidean 3-space V, our τ is a function on M, constant in the direction of the N factor, and, as a function $\tau : S^2 \to \mathbf{R}$, it is the restriction to S^2 of a nonzero linear homogeneous function $V \to \mathbf{R}$.

The quadruple (M,g,m,τ) then satisfies (0.1) or (0.2); see [11, §30] or [10, §25].

Remark 43.1. Unlike the examples just described, any quadruple (M,g,m,τ) constructed as in §3 is *locally irreducible* in the sense that no open submanifold of (M,g) is biholomorphically isometric to a Cartesian product of lower-dimensional Kähler manifolds.

In fact, the horizontal and vertical distributions on M then consist of eigenvectors of both the Ricci tensor r of g and the second covariant derivative $\nabla d\tau$ of τ relative to g, with some eigenvalue functions λ, μ for r and ϕ, ψ for $\nabla d\tau$, all of which are also functions of t, that is, of τ. (We borrow these notations from [10, formula (7.4)], cf. [10, §8].) One then has $Q = 2(\tau - c)\phi$ and $(\lambda - \mu)\tau = 2(m-1)(\tau - c)\, d\phi/d\tau$. Thus, by (2.5.a,b,c), ϕ is not constant on any nonempty open subset of M, and so g, restricted to any such subset, cannot be Einstein. Local irreducibility of (M,g) now follows from [10, Corollaries 13.2(iii) and 9.3].

44. Bérard Bergery's and Page's examples

Given an integer $m \geqslant 2$, let $\mathcal{S}_m^{\mathsf{I}}$ be the set of all p-rational points in the I component of the moduli curve (cf. Definition 1.1 and the lines following (1.1)). By applying the construction of §1 to all points $(u,v) \in \mathcal{S}_m^{\mathsf{I}}$ and appropriate additional data, we obtain the *second family* of quadruples (M,g,m,τ) satisfying (0.1) or (0.2). Since u, v then are both positive, so is the function $t = \tau/c$ on M (cf. Remark 3.2); thus, all compact Kähler manifolds (M,g) obtained here are *globally* conformally Einstein.

The second family has been known for over two decades: the (essentially unique) quadruple with (0.2) was found by Page [14], and those with (0.1) by Bérard Bergery [4]. More precisely, they both described the corresponding conformally related Einstein manifolds. (See also [10, §26].)

Proof of Theorem 1.5, parts (a), (d). Assertion (d), which also implies finiteness of $\mathcal{S}_m^{\mathsf{I}}$, is obvious from (e) in §38 and Definition 1.1, while (a) in Theorem 1.5 follows from (d), Lemma 42.1(a) for $\ell = 1$, and Remark 42.2. $\square$

Of particular interest is the case where $m = 2$, with (M,g) conformal to the Page manifold. By Theorem 1.5(d), $\mathcal{S}_2^{\mathsf{I}}$ has a single element (u,v) with $v = u/(u-1)$. Explicitly, u then is given by $u = [(3-\alpha+2\beta)^{1/2} + (3+\alpha)^{1/2}](\alpha+\beta)/6$, where $\alpha = (13+\sqrt{142}\,)^{1/3} + (13-\sqrt{142}\,)^{1/3} - 1$ and $\beta = (6+\alpha^2)^{1/2}$, so that α is the unique real root of the equation $\alpha^3 + 3\alpha^2 - 6\alpha = 34$. (In fact, relation $P(u) = 1/2$, with P as in (34.1) for $m = 3$, amounts to $(u-2)(3u^4 - 8u^3 + 6u^2 - 2) = 0$.) Approximately, $u \approx 1.560$ and $v \approx 2.786$.

45. A third family of examples

The *third family* of quadruples (M,g,m,τ) satisfying (0.1) or (0.2) is obtained by applying the construction preceding Proposition 1.2 to p-rational points of the moduli curve $\mathcal{C}$ that lie, for even m, in $! \smallsetminus \{(0,0)\}$, or, for odd m, in the union of $! \smallsetminus \{(0,0)\}$ and the $\mathcal{H}$-beam $\mathsf{X} \cap \mathcal{H}$ of X (see the lines following (1.1)). The second and third families may be thought of as related to each other by a form of analytic continuation, since they both use p-rational points (u,v) lying on the hyperbola $\mathcal{H}$ given by $uv = u + v$ (cf. §49).

In the complex dimension $m = 2$, quadruples of the third family were first found by Hwang and Simanca [15]) and Tønnesen-Friedman [18].

Just as we did for the second family in §44, we will now obtain a rough idea about the "size" of the third family by estimating the number of p-rational points involved. In §44 that amounted to proving (a) and (d) in Theorem 1.5. Here, the corresponding results consist of Theorem 1.6, already established in §38, and assertions (b), (e) in Theorem 1.5, proved at the end of this section. (The other parts of Theorem 1.5 pertaining to the third family then are immediate from (b) in §38: $\mathcal{S}_m^{\mathcal{H}}$ is finite since a bounded interval contains only finitely many rational numbers that can be written as fractions with a denominator in $\{1,\ldots,m\}$, while p maps $\mathcal{S}_m^{\mathcal{H}}$ into $(-2,0)$ since $P(u_+^*) > -2$, as shown in (iii) below.)

Let P, u_+, x, x_∞ and z depend on an odd integer $m \geqslant 3$ as in (34.1), (36.2), Lemma 25.2(b), Remark 40.3 and §21, and let $t^* = t/(t-1)$ if $t \in \mathbf{R} \smallsetminus \{1\}$. Then

i) $x_\infty < u_+^* < x$, ii) $u_+^* \to x_\infty$ as $m \to \infty$, iii) $P(u_+^*) > -2$, iv) $P(u_+^*) \to -\sqrt{2}$ as $m \to \infty$,

v) $z/(2-z)$ is a decreasing function of the odd integer $m \geqslant 3$, such that $z/(2-z) > -1 + 1/m$ and $z/(2-z) \to -1/\sqrt{2}$ as $m \to \infty$,

vi) $x^* < u_+ < x_\infty^* < 1$ and $P(u_+) < 2$, which improves on (36.2),

vii) $3/2 < P_+(2/3) < 2$, with $P_\pm$ as in §37, viii) $(x_\infty - 2)/x_\infty = \sqrt{2}$.

In fact, (v) follows from Remark 40.3: $z/(2-z) = -1 + 2/(2-z) > -1 + 1/m$ since $z > 2(1-m)$, which for $m \geqslant 5$ is clear as $z > x > x_\infty$ and so $z > x_\infty > 2(1-m)$, and for $m = 3$ is obvious as $z \approx -1.3$ (see the end of §21).

Let us now set $u = x_\infty$. We obviously have $(u-2)/u = \sqrt{2}$, that is, (viii), and $u - 1 = -u^2/4$. Dividing (34.1) by $-u^2$ and using the definition of R_m preceding (40.1), we now get $-\sqrt{2}\,P(u) = [1 - 1/(2m)][1 - 1/R_m(u)] + 1$, for any fixed odd integer $m \geqslant 3$. Relation $R_m(u) \geqslant 3m$ for any odd $m \geqslant 1$, easily obtained from (40.1.ii) with $t = u = x_\infty$ and $\sigma = 4$ using induction on $m = 1, 3, 5, \ldots$, now yields $P(u) \leqslant P_* < 0$ for P_* given by $-6\sqrt{2}m^2 P_* = 12m^2 - 5m + 1$. On the other hand, dividing (35.1) by u and using (viii) we obtain $(u-1)\,\dot{P} = mP^2 + \sqrt{2}(m-1)P - 1$, the right-hand side of which is easily verified to be positive when P (that is, $P(u)$) is replaced by P_* and $m \geqslant 3$. As $-1 < 0$, that right-hand side is a quadratic function of P with roots of opposite signs; thus, it is strictly decreasing on the subset of the negative P axis on which it is positive. Therefore it is positive at $P(u)$ as well, and so $(u-1)\,\dot{P} > 0$, at $u = x_\infty$, that is, $\dot{P}(u) < 0$. Since we also have $P(u) < 0$, Remark 36.1 now gives $u < u_+^*$ for $u = x_\infty$, and, as $x^* < u_+$ (see (36.2)), we obtain (i) and (ii) from Remark 40.3.

Next, let $P_\pm$ be as in §37, so that $P_\pm$ depend here on an odd integer $m \geqslant 3$, and let a sequence $u_m < 0$, $m = 3, 5, 7, \ldots$, converge to a limit $u_\infty < 0$ as $m \to \infty$. The expression for $P_-(u)$ with any $u < 0$, provided by the quadratic formula, shows that $P_-(u_m)$ then has the limit $-(u_\infty - 2)/u_\infty$, equal to $-\sqrt{2}$ if $u_\infty = x_\infty$ (see (viii)). Using the sequence $u_m = u_+^*$ and (ii), we now get (iv).

For an odd integer $m \geqslant 3$, let $\mathcal{F}(P) = mP^2 - 2(m-1)P - 1$, which is $3/2$ times the right-hand side of (35.1) with $u = 2/3$. As $\mathcal{F}(3/2) < 0 < \mathcal{F}(2)$, (vii) follows, $P = P_+(2/3)$ being the positive root of $\mathcal{F}$. Next, for any odd $m \geqslant 5$ (or, $m = 3$), $1 + \varSigma(0)/G(u)$ in (34.1) with $u = -2$ is negative

(or, respectively, equal to $22/49$). This is clear from the definition of R_m, since $R_3(-2) = 49/27$ by (40.1.iii), from which, using (40.1.ii) and induction on k, we obtain $0 < R_k(-2) < 1$ for every integer $k \geqslant 4$. Hence, if $m \geqslant 5$ (or, $m = 3$), (34.1) gives $P(-2) > -1/2$ (or, $P(-2) = -52/49$), so that, (d) in §35 with $u = -2$ yields $P(2/3) < 1/2$ (or, respectively, $P(2/3) = 52/49$). When $m > 3$ this clearly yields $u_+ > 2/3$, since $2/3 \notin [u_+, 1]$, as $P \geqslant 1$ on $[u_+, 1]$ by (36.3); if, however, $m = 3$, relations $P_+(2/3) \in (3/2, 2)$ (see (vii)) and $0 < P(2/3) = 52/49 < 3/2$ give, by (i) in §37, $P_-(u) < P(u) < P_+(u)$ with $u = 2/3$, so that $\dot P(2/3) > 0$ (see (iii) in §37), and hence $u_+ > 2/3$, as $\dot P \leqslant 0$ on $[u_+, 1]$ by (36.3). Thus, $P(u_+) < 2$, which we verify in three steps. First, $P(u_+) = P_+(u_+)$ by (iv),(i) in §37, since $P(u_+)$ is the maximum of P on $(0, 1)$. Secondly, $P_+(u_+) < P_+(2/3)$, since we just showed that $u_+ > 2/3$, and P_+ is decreasing on $(0, 1)$, cf. (37.1). Thirdly, $P_+(2/3) < 2$ by (vii). Now (iii) follows from (d) in §35, while (d),(a) in §35 and (iii),(i) give (vi).

Proof of Theorem 1.5, parts (b), (e). Assertion (e) is immediate from (b) in §38, since (iii) and (v) above give $-2 < P(u_+^*) < -1$ and $z/(2-z) > -1 + 1/m$. To prove (b), recall (§1) that the $\mathcal{H}$-beam of X is the graph of the function $v = u^*$ on the interval $(-\infty, z)$ of the variable u. Dividing $(-\infty, z)$ into the three subintervals $(-\infty, u_+^*]$, $[u_+^*, u']$, $[u', z]$, for the unique $u' < 0$ with $P(u') = -1$ (cf. Remark 36.1), we also divide the $\mathcal{H}$-beam into three segments (subbeams). Since P is the restriction of the function p to the hyperbola $v = u^*$ (see §34), Remark 36.1 shows that p maps the set of all p-rational points in the first (or, second, or, third) subbeam bijectively onto the set of all rational numbers that have positive denominators not exceeding m and lie in an interval with the endpoints $P(u_+^*)$ and -1 (or, again, $P(u_+^*)$ and -1, or, respectively, -1 and $z/(2-z)$). Using, instead of p, the function $1 + 1/p$ for the first two subbeams, and $1 + p$ for the third, we obtain an analogous property for new intervals, with the lower endpoint 0 and the upper endpoint $1 + 1/P(u_+^*)$ for the first two, or $1 + z/(2-z)$ for the third subbeam. Now (b) in Theorem 1.5 is immediate from Lemma 42.1(b) (cf. Remark 42.2), where, for each subbeam, $\ell_\infty = 1 - 1/\sqrt{2}$. (See (iv) and (v) above.) This completes the proof. ∎

Remark 45.1. According to the preceding three lines, asymptotically, the three subbeams contribute the same number of p-rational points: the share of each subbeam, divided by m^2, has the limit $3\sqrt{2}(\sqrt{2}+1)/\pi^2 \approx 1.038$ as $m \to \infty$.

Remark 45.2. If $m \geqslant 2$ is even, *every* compact Kähler-Einstein manifold (N, h) of the odd complex dimension $m - 1$ appears as an ingredient of the construction of some quadruple (M, g, m, τ) of the third family, except for one restriction: if h is Ricci-flat, its Kähler cohomology class must be a real multiple of an integral class. This is immediate is one combines Theorem 1.6(ii) with the definition of δ in §1 (second paragraph after (1.1)), relations (1.2.ii) and $\delta = \operatorname{sgn} \kappa$ in §1, and the third paragraph of Remark 4.3.

46. The fourth family: examples of a new type

Given an odd integer $m \geqslant 3$, let $\mathcal{S}_m^{\mathcal{T}}$ be the set of those p-rational points in the $\mathcal{T}$-beam $\mathsf{X} \cap \mathcal{T}$ of the X component of the moduli curve (§1) which do not lie in the $\mathcal{H}$-beam $\mathsf{X} \cap \mathcal{H}$. Our *fourth family* of quadruples (M, g, m, τ) with (0.1) or (0.2) is obtained from the construction of §1 applied to points $(u, v) \in \mathcal{S}_m^{\mathcal{T}}$. (See also Remark 46.1.)

The fourth family exists only in the odd complex dimensions $m \geqslant 9$, since, according to Theorem 1.5(f) (proved below), $\mathcal{S}_m^{\mathcal{T}}$ is empty for $m = 3, 5, 7$. Also, for ℓ_m and $\mathcal{P}(\ell_m, m)$ as in §41 and Lemma 42.1, and with $|\,|$ denoting cardinality,

$$|\mathcal{S}_m^{\mathcal{T}}| = 2|\mathcal{P}_m|, \qquad \text{where} \quad \mathcal{P}_m = \mathcal{P}(\ell_m, m). \tag{46.1}$$

(As $\ell_m > 1$, we thus have $|\mathcal{S}^{\mathcal{T}}_m| \leqslant m(m-1)$.) In fact, we can define a two-to-one surjective mapping $\mathcal{S}^{\mathcal{T}}_m \ni (u,v) \mapsto (k,d) \in \mathcal{P}_m$ by assigning to (u,v) the pair (k,d) of relatively prime positive integers such that $(k-d)/d$ is the value at (u,v) of the function p described in §34. (This, including the fact that $(k,d) \in \mathcal{P}(\ell_m, m)$, is clear from (a) in §38 and Definition 1.1.)

Proof of Theorem 1.5, parts (c), (f)*.* First, (46.1), (e) in §41 and Lemma 42.1(b) give (c).

Next, let us recall that $\mathcal{P}_m$ in (46.1) is the set of all relatively prime integer pairs (k,d) with $1 \leqslant k\ell_m < d \leqslant m$. Since that gives $\ell_m \leqslant k\ell_m < m$, no such pair exists if $\ell_m > m$. Thus, (c) in §41 and (46.1) show that $\mathcal{S}^{\mathcal{T}}_m = \emptyset$ for $m = 3,5,7$. However, if $\ell_m < m$, the set $\mathcal{P}_m$ is nonempty, as $(1,m) \in \mathcal{P}_m$. Therefore, $|\mathcal{S}^{\mathcal{T}}_m| = 2|\mathcal{P}_m| \geqslant 2$ for any odd $m \geqslant 9$, in view of (46.1) and (b) in §41. If m is odd and $m \geqslant 19$, we have $\ell_m < 18$ (see (a) in §41), so that $\mathcal{P}_m$ contains the $(m-17)$-element subset of all (k,d) with $k = 1$ and $18 \leqslant d \leqslant m$, which yields $|\mathcal{S}^{\mathcal{T}}_m| \geqslant 2(m-17)$ by (46.1).

Finally, $8 < \ell_9 < 9$ by (c) in §41. Thus, $(k,d) = (1,9)$ is the only integer pair for which $1 \leqslant k\ell_9 < d \leqslant 9$, with $m = 9$ (since that gives $8k < k\ell_9 < 9$, and so $k = 1$). From (46.1) we now get $|\mathcal{S}^{\mathcal{T}}_9| = 2|\mathcal{P}_9| = 2$, completing the proof. ∎

Note that, as $9 < \ell_{11} < 10$ (cf. (c) in §41), the same argument as in the last three lines gives $|\mathcal{S}^{\mathcal{T}}_{11}| = 4$, since $\mathcal{P}_{11}$ has just two elements: $(1,10)$ amd $(1,11)$.

Remark 46.1*.* By Lemma 25.2(b), $(\mathsf{X} \cap \mathcal{T}) \cap (\mathsf{X} \cap \mathcal{H}) = \{(x,x^*)\}$, so that $\mathcal{S}^{\mathcal{T}}_m$, defined at the beginning of this section, is also the set of all p-rational points in $\mathsf{X} \cap \mathcal{T}$ *other than* (x,x^*).

For some odd integers $m \geqslant 3$ the phrase 'other than (x,x^*)' used here is redundant, since (x,x^*) is not p-rational. Actually, we do not know if (x,x^*) can be p-rational *for any* odd $m \geqslant 3$.

However, if $m \geqslant 3$ is odd and $2\ell_m > m$, then p-rationality of (x,x^*) implies that ℓ_m is an *integer* and $\ell_m \leqslant m$. In fact, by (a) in §38, $q_* = -1 + 1/\ell_m$ is the value of p at (x,x^*), and, as $q_* \in (-1,0)$ (Remark 27.2), the number q_*, now assumed rational, must have the form $(k-d)/d$ for some $k,d \in \mathbf{Z}$ with $1 \leqslant k < d \leqslant m$ (cf. Definition 1.1 and (a) in §38); hence $0 < k\ell_m = d \leqslant m$, so that $k = 1$ (as $k\ell_m \leqslant m < 2\ell_m$) and $\ell_m = d \in \mathbf{Z}$.

For instance, (x,x^*) is *not* p-rational for any $m \in \{3,5,7,9,11\}$. Namely, (c) in §41 then gives $2\ell_m > m$ (also for $m = 13,15,17$, as $2\ell_m > 2\ell_{11} > 18 > m$ by (e) in §41). However, again by (c) in §41, one of the two conditions just named, necessary for p-rationality of (x,x^*), fails: $\ell_m > m$ if $m \in \{3,5,7\}$, and $\ell_9, \ell_{11} \notin \mathbf{Z}$. (One can extend this argument and conclusion to $m \in \{13,15,\dots,23\}$, since a numerical approximation of ℓ_m then gives $2\ell_m > m$ and $\ell_m \notin \mathbf{Z}$.)

47. Appendix: a version of Mertens's theorem

Assertion (a) in Lemma 42.1 will now be derived using a slightly modified version of a standard proof of Mertens's theorem (cf. [8, p. 59]). Here $m \geqslant 1$ is treated as a real variable, even though in Lemma 42.1(a) it stands for an odd integer with $m \geqslant 3$.

Given a real number $\ell \geqslant 1$, we define $W_\ell(m)$ to be the set of all pairs (k,d) of integers with $1 \leqslant k\ell < d \leqslant m$. If, in addition, n is an integer with $1 \leqslant n \leqslant m$, let $W_\ell(m,n) \subset W_\ell(m)$ consist of those $(k,d) \in W_\ell(m)$ in which both k and d are divisible by n. Obviously, for positive integers $n_1,\dots,n_s$ that are pairwise relatively prime, and with $|\,|$ denoting cardinality,

$$\text{a)}\ \ W_\ell(m,\prod_{j=1}^{s} n_j) \,=\, \bigcap_{j=1}^{s} W_\ell(m,n_j), \qquad \text{b)}\ \ |W_\ell(m,n)| \,=\, |W_\ell(m/n)|, \tag{47.1}$$

where (b) is due to the bijection $W_\ell(m,n) \to W_\ell(m/n)$ given by $(k,d) \mapsto (k/n, d/n)$. Next, given $\ell, m \in [1,\infty)$, we let $\mathcal{P}(\ell,m)$ denote, as in Lemma 42.1, the subset of $W_\ell(m)$ formed by those integer pairs (k,d) which, in addition to having $1 \leqslant k\ell < d \leqslant m$, are also *relatively prime.*

For any finite family $\mathcal{A}$ of finite sets, induction on its cardinality $|\mathcal{A}|$ easily implies that $|\bigcup\mathcal{A}| = \sum_{\mathcal{B}}(-1)^{|\mathcal{B}|-1}\,|\bigcap\mathcal{B}|$, with summation over all nonempty subfamilies $\mathcal{B}$ of $\mathcal{A}$. If $\mathcal{A}$ consists of all sets $W_\ell(m,p)$, where $m\in[1,\infty)$ is fixed and p runs through all primes with $p\leqslant m$, then $\bigcup\mathcal{A}$ clearly coincides with $W_\ell(m)\smallsetminus\mathcal{P}(\ell,m)$, so that (47.1) and our formula for $|\bigcup\mathcal{A}|$ give

$$|\mathcal{P}(\ell,m)| \quad = \sum_{1\leqslant j\leqslant m}\mu_j\,|W_\ell(m/j)|\,. \tag{47.2}$$

Here μ is the *Möbius function* [8]), assigning to every integer $j\geqslant 1$ the value μ_j with $\mu_j=0$ if j is divisible by the square of a prime and $\mu_j=(-1)^k$ when j is the product of k distinct primes, for $k\geqslant 0$. (Thus, $\mu_1=1$.) The summation index j, here and below, is an integer.

Setting $f(m)=2\ell|W_\ell(m)|-m^2$, we have $|f(m)|\leqslant\alpha m$ for every $m\in[1,\infty)$, with the constant $\alpha=2\ell+2$. In fact, given $m\in[1,\infty)$, let n (or, r) be the largest integer not exceeding m (or, m/ℓ) and, for any $k\in\{1,\dots,r\}$, let s_k be the largest integer with $s_k\leqslant k\ell$. Thus, $W_\ell(m)$ contains exactly $n-s_k$ points with the first coordinate k, namely, (k,d) with $s_k<d\leqslant n$. Consequently, $|W_\ell(m)|=\sum_{k=1}^r(n-s_k)$, and so the inequalities $k\ell-1<s_k\leqslant k\ell$, that is, $n-k\ell\leqslant n-s_k<n+1-k\ell$, yield $[2n-(r+1)\ell]r\leqslant 2|W_\ell(m)|\leqslant[2(n+1)-(r+1)\ell]r$. Hence $-(2\ell+2)m\leqslant f(m)\leqslant 2m$. Namely, $-m-\ell\leqslant-(r+1)\ell\leqslant-m$ and $m-1\leqslant n\leqslant m$ due to our choice of m and r, so that $2n-(r+1)\ell\geqslant m-\ell-2$ and $2(n+1)-(r+1)\ell\leqslant m+2$, while $-1+m/\ell\leqslant r\leqslant m/\ell$. Thus, $-\alpha m\leqslant f(m)\leqslant\alpha m$.

Given a bounded sequence $\mu_1,\mu_2,\dots$ of real numbers and a function f of the real variable $m\geqslant 1$ such that $|f(m)|\leqslant\alpha m$ for all m and some constant $\alpha\geqslant 0$, we necessarily have $m^{-2}\sum_{1\leqslant j\leqslant m}\mu_j f(m/j)\to 0$ as $m\to\infty$, which is obvious since $\sum_{1\leqslant j\leqslant m}j^{-1}\leqslant 1+\log m$ due to a standard area-under-the-graph estimate. For f and α as in the last paragraph, with the Möbius function μ, this shows that, by (47.2), if $\sum_{1\leqslant j\leqslant m}(\mu_j/j^2)$ has a limit as $m\to\infty$, then so does $2\ell|\mathcal{P}(\ell,m)|/m^2$, and the limits coincide. However, $\sum_{1\leqslant j\leqslant m}(\mu_j/j^2)$ clearly does converge, as $m\to\infty$, to the product $\prod_p(1-1/p^2)$, where p runs over all primes; now (a) in Lemma 42.1 follows since the inverse of the product equals $\sum_{n=1}^\infty n^{-2}=\pi^2/6$ due to a special case of *Euler's identity*, obtained by expanding each factor $1/(1-1/p^2)$ into a geometric series. (Cf. [8, pp. 61 and 76].)

48. Appendix: decimal approximations

We show here that, for χ_m defined in Lemma 40.2,

$$\chi_7>1.9,\quad 2.2<\chi_9<2.3<2.4<\chi_{11}<2.5,\quad \chi_{13}<2.6,\quad \chi_{15}<2.7,\quad \chi_{17}<2.8. \tag{48.1}$$

This is verified using the following algorithm, designed for calculations that can even be done by hand, and produce upper/lower bounds on χ_m having the form $\chi_m<\sigma_m$ or $\chi_m>\sigma_m$ for a fixed odd integer $m\geqslant 3$ and a rational number $\sigma_m>0$ with a simple decimal expansion.

The values of $\Sigma_j(0)$ for $j=1,\dots,m-1$ can easily be found from (40.1.i); when $m=17$, they are 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845. In formula (40.1.iv), with m replaced by $m-1$, we now estimate each term $(-1)^j\Sigma_{m-j-1}(0)\sigma^j$ for $\sigma=\sigma_m$ from above/below by the nearest integer. When these estimates are added up, (40.1.iv) yields an upper/lower bound on $\Sigma_{m-1}(0)R_{m-1}(t_m)$, for t_m related to σ_m as in (40.2).

Whenever this last bound happens to ensure negativity/positivity of $R_{m-1}(t_m)+K_m(\sigma_m)$, Remark 40.1(b) will imply negativity/positivity of $\chi_m-\sigma_m$, as required.

Specifically, let $m\in\{7,9,11\}$. If $\sigma_7=1.9$, $\sigma_9=2.2$, $\sigma_{11}=2.4$, these steps give $42R_6(t_7)>42-27+18-14+13-25-24=-17$, $429R_8(t_9)>429-291+203-150+117-104+113-250-275=-208$, $4862R_{10}(t_{11})>4862-3432+2471-1825+1393-1115+955-918+1100-2642-3171=-2322$, so that $R_{m-1}(t_m)>-1/2$ for $m=7,9,11$, while $K_7(\sigma_7)>11/19>1/2$ as $\sigma_7<2$,

$K_9(\sigma_9) > 5/9 > 1/2$ as $\sigma_9 < 2.5$, and $K_{11}(\sigma_{11}) = 19/33 > 1/2$. Hence $R_{m-1}(t_m) + K_m(\sigma_m) > 0$ and Remark 40.1(b) yields the lower bounds on $\chi_7, \chi_9, \chi_{11}$ in (48.1).

Next, for $m = 9, 11, 13, 15, 17$, let us set, this time, $\sigma_9 = 2.3$, $\sigma_{11} = 2.5$, $\sigma_{13} = 2.6$, $\sigma_{15} = 2.7$, $\sigma_{17} = 2.8$. Our algorithm now yields $429R_8(t_9) < 429-303+223-170+140-128+149-340-391 = -391$, $4862R_{10}(t_{11}) < 4862-3575+2682-2062+1641-1367+1221-1220+1526-3814-4768 = -4874$, $58786R_{12}(t_{13}) < 58786-43669+32868-25133+19605-15683+12975-11244+10442-10859+14117-36703-47714 = -42212$, $742900R_{14}(t_{15}) < 742900-561632+428550-330595+258387-205189+166204-138076+118621-106758+102946-111181+150095-405255-547094 = -438077$, $9694845R_{16}(t_{17}) < 9694845 - 7488432 + 5824336 - 4566279 + 3613317 - 2890653 + 2342951 - 1929488 + 1620771 - 1396356 + 1244027 - 1161091 + 1161092 - 1300422 + 1820592 - 5097655-7136717 = -5645162$, so that $R_8(t_9) < -0.9$, $R_{10}(t_{11}) < -1$ and $R_{m-1}(t_m) < -0.58$ for $m = 13, 15, 17$, while $K_9(\sigma_9) < 3/5 = 0.6$ as $\sigma_9 > 2$, $K_{11}(\sigma_{11}) = 38/67 < 1$, and $K_m(\sigma_m) < 0.57$ for $m = 13, 15, 17$. Thus, $R_{m-1}(t_m) + K_m(\sigma_m) < 0$ for $m = 9, 11, 13, 15, 17$, and Remark 40.1(b) gives the upper bounds on χ_m required in (48.1).

49. Appendix: vertical compactness

The following discussion provides a differential-geometric background for the moduli curve, and is not used to derive any results of this paper. To save space, our presentation is brief.

Let a quadruple (M, g, m, τ) have all the properties listed in (0.1) or (0.2) except for compactness of M. The set $M' \subset M$ on which $d\tau \neq 0$ then is open and dense in M (cf. [10, Remark 5.4], while the complex vector subbundle of TM' spanned by the g-gradient $\nabla\tau$ is an integrable real 2-dimensional distribution on M' with totally geodesic leaves [10, the end of §7]. We will say that the quadruple (M, g, m, τ) satisfies the *vertically compact version* of (0.1) or (0.2) if every such leaf is contained in a compact submanifold of real dimension 2 in M.

The constructions of §3 and §43 applied to data satisfying all the assumptions stated there except for compactness of N always leads to (M, g, m, τ) satisfying the vertically compact version of (0.1) or (0.2). (In fact, the compactness assumption is never used in either construction.) Similarly, the vertically compact version of (0.1) or (0.2) holds for (M, g, m, τ) obtained as in §1 from (u, v) and additional data having all the properties listed in §1 except that (u, v) is assumed merely to lie in the moduli curve $\mathcal{C}$ (and is not required to be p-rational), while N is not necessarily compact.

Conversely, every quadruple (M, g, m, τ) satisfying the vertically compact version of (0.1) or (0.2) is obtained in this way from some $(u, v) \in \mathcal{C}$ and additional data mentioned above. In fact, Theorems 33.2, 33.3 and 34.3 in [11] (which involve the four types mentioned in (i) - (iii) of §39 of this paper), as well as Lemma 39.2 in §39, all remain valid, with essentially the same proofs, also in the vertically compact case, and so our claim follows as in the last three lines of §39.

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